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Fixed points of coprime operator groups

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A B S T R A C T

Let m be a positive integer and A an elementary abelian group of order q^r with $r \geqslant 2$ acting on a finite q'-group G. We show that if for some integer d such that $2^d \leqslant r-1$ the dth derived group of $C_G(a)$ has exponent dividing m for any $a \in A^{\#}$, then $G^{(d)}$ has $\{m,q,r\}$-bounded exponent and if $\gamma_{r-1}(C_G(a))$ has exponent dividing m for any $a \in A^{\#}$, then $\gamma_{r-1}(G)$ has $\{m,q,r\}$-bounded exponent.

1. Introduction

Let A be a finite group acting coprimely on a finite group G. It is well known that the structure of the centralizer $C_G(A)$ (the fixed-point subgroup) of A has strong influence over the structure of G. To exemplify this we mention the following results.

The celebrated theorem of Thompson [18] says that if A is of prime order and $C_G(A) = 1$, then G is nilpotent. On the other hand, any nilpotent group admitting a fixed-point-free automorphism of prime order q has nilpotency class bounded by some function $h(q)$ depending on q alone. This result is due to Higman [6]. The reader can find in [8] and [9] an account on the more recent developments related to these results. The next result is a consequence of the classification of finite simple groups [21]: If A is a group of automorphisms of G whose order is coprime to that of G and $C_G(A)$ is nilpotent or has

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odd order, then G is soluble. Once the group G is known to be soluble, there is a wealth of results bounding the Fitting height of G in terms of the order of A and the Fitting height of $C_G(A)$. This direction of research was started by Thompson in [19]. The proofs mostly use representation theory in the spirit of the Hall–Higman work [5]. A general discussion of these methods and their use in numerous fixed-point theorems can be found in Turull [20].

Following the solution of the restricted Burnside problem it was discovered that the exponent of $C_G(A)$ may have strong impact over the exponent of G. Remind that a group G is said to have exponent m if $x^m = 1$ for every $x \in G$ and m is the minimal positive integer with this property. The next theorem was obtained in [10].

Theorem 1.1. *Let q be a prime, m a positive integer and A an elementary abelian group of order q^2. Suppose that A acts as a coprime group of automorphisms on a finite group G and assume that $C_G(a)$ has exponent dividing m for each $a \in A^\#$. Then the exponent of G is $\{m, q\}$-bounded.*

Here and throughout the paper $A^\#$ denotes the set of nontrivial elements of A. The proof of the above result involves a number of deep ideas. In particular, Zelmanov's techniques that led to the solution of the restricted Burnside problem [24] are combined with the Lubotzky–Mann theory of powerful p-groups [13], Lazard's criterion for a pro-p group to be p-adic analytic [11], and a theorem of Bakhturin and Zaicev on Lie algebras admitting a group of automorphisms whose fixed-point subalgebra is PI [1].

Another quantitative result of similar nature was proved in the paper of Guralnick and the second author [4].

Theorem 1.2. *Let q be a prime, m a positive integer. Let G be a finite q'-group acted on by an elementary abelian group A of order q^3. Assume that $C_G(a)$ has derived group of exponent dividing m for each $a \in A^\#$. Then the exponent of G' is $\{m, q\}$-bounded.*

Note that the assumption that $|A| = q^3$ is essential here and the theorem fails if $|A| = q^2$. The proof of Theorem 1.2 depends on the classification of finite simple groups.

It was natural to expect that Theorems 1.1 and 1.2 admit a common generalization that would show that both theorems are part of a more general phenomenon. Let us denote by $\gamma_i(H)$ the ith term of the lower central series of a group H and by $H^{(i)}$ the ith term of the derived series of H. The following conjecture was made in [17].

Conjecture 1.3. *Let q be a prime, m a positive integer and A an elementary abelian group of order q^r with $r \geqslant 2$ acting on a finite q'-group G.*

(1) *If $\gamma_{r-1}(C_G(a))$ has exponent dividing m for any $a \in A^\#$, then $\gamma_{r-1}(G)$ has $\{m, q, r\}$-bounded exponent*;
(2) *If, for some integer d such that $2^d \leqslant r - 1$, the dth derived group of $C_G(a)$ has exponent dividing m for any $a \in A^\#$, then the dth derived group $G^{(d)}$ has $\{m, q, r\}$-bounded exponent.*

The main purpose of the present paper is to confirm Conjecture 1.3. Theorem 6.1 and Theorem 7.4 show that both parts of the conjecture are correct. The main novelty of the paper is the introduction of the concept of A-special subgroups of G (see Section 3). Using the classification of finite simple groups it is shown in Section 4 that the A-invariant Sylow p-subgroups of $G^{(d)}$ are generated by their intersections with A-special subgroups of degree d. This enables us to reduce the proof of Conjecture 1.3 to the case where G is a p-group, which can be treated via Lie methods. The idea of this kind of reduction has been anticipated already in [4]. In Section 6 we give a detailed proof of part (2) of Conjecture 1.3. In Section 7 we briefly describe how the developed techniques can be used to prove part (1) of Conjecture 1.3.

Throughout the article we use the term "$\{a, b, c, \ldots\}$-bounded" to mean "bounded from above by some function depending only on the parameters $a, b, c, \ldots$".

2. Preliminary results

We start with the following elementary lemma.

Lemma 2.1. *Suppose that a nilpotent group G is generated by subgroups $G_1, \dots, G_t$ such that $\gamma_i(G) = \langle \gamma_i(G) \cap G_j \mid 1 \leqslant j \leqslant t \rangle$ for all $i \geqslant 1$. Then $G = G_1 G_2 \cdots G_t$.*

Proof. We argue by induction on the nilpotency class c of G. If $c = 1$, then G is abelian and the result is clear. Assume that $c \geqslant 2$. Let $K = \gamma_c(G)$. Since K is central, it is abelian and we have $K = K_1 K_2 \cdots K_t$, where $K_j = K \cap G_j$ for $j = 1, \dots, t$. By induction we have

$$G = G_1 G_2 \cdots G_t K = G_1 G_2 \cdots G_t K_1 K_2 \cdots K_t.$$

Since each subgroup K_j is central in G and $K_j \leqslant G_j$, it follows that $G = G_1 G_2 \cdots G_t$, as required. □

We now collect some facts about coprime automorphisms of finite groups. The two following lemmas are well known (see [3, 5.3.16, 6.2.2, 6.2.4]).

Lemma 2.2. *Let A be a group of automorphisms of the finite group G with $(|A|, |G|) = 1$.*

(1) *If N is an A-invariant normal subgroup of G, then $C_{G/N}(A) = C_G(A)N/N$;*
(2) *If H is any A-invariant p-subgroup of G, then H is contained in an A-invariant Sylow p-subgroup of G.*

Lemma 2.3. *Let q be a prime, G a finite q'-group acted on by an elementary abelian q-group A of rank at least 2. Let $A_1, \dots, A_s$ be the maximal subgroups of A. If H is an A-invariant subgroup of G we have $H = \langle C_H(A_1), \dots, C_H(A_s) \rangle$. Furthermore if H is nilpotent then $H = \prod_i C_H(A_i)$.*

We also need the following result, which is a well-known corollary of the classification of finite simple groups.

Lemma 2.4. *Let G be a finite simple group and A a group of automorphisms of G with $(|A|, |G|) = 1$. Then A is cyclic.*

We conclude this section by citing an important theorem due to Gaschütz. The proof can be found in [7, p. 121] or in [14, p. 191].

Theorem 2.5. *Let N be a normal abelian p-subgroup of a finite group G and let P be a Sylow p-subgroup of G. Then N has a complement in G if and only if N has a complement in P.*

3. *A*-special subgroups

In this section we introduce the concept of *A-special subgroups* of G. For every integer $k \geqslant 0$ we define A-special subgroups of G of degree k in the following way.

Definition 3.1. Let q be a prime and A an elementary abelian q-group acting on a finite q'-group G. Let $A_1, \dots, A_s$ be the subgroups of index q in A and H a subgroup of G.

- We say that H is an A-special subgroup of G of degree 0 if and only if $H = C_G(A_i)$ for suitable $i \leqslant s$.
- Suppose that $k \geqslant 1$ and the A-special subgroups of G of degree $k-1$ are defined. Then H is an A-special subgroup of G of degree k if and only if there exist A-special subgroups J_1, J_2 of G of degree $k-1$ such that $H = [J_1, J_2] \cap C_G(A_j)$ for suitable $j \leqslant s$.

Here as usual $[J_1, J_2]$ denotes the subgroup generated by all commutators $[x, y]$ where $x \in J_1$ and $y \in J_2$. We note that all A-special subgroups of G of any degree are A-invariant. Assume that A has order q^r. It is clear that for a given integer k the number of A-special subgroups of G of degree k is $\{q, r, k\}$-bounded. Let us denote this number by s_k.

The A-special subgroups have certain properties that will be crucial for the proof of the main result of this paper.

Proposition 3.2. *Let A be an elementary abelian q-group of order q^r with $r \geqslant 2$ acting on a finite q'-group G and let $A_1, \dots, A_s$ be the maximal subgroups of A. Let $k \geqslant 0$ be an integer.*

(1) *If $k \geqslant 1$, then every A-special subgroup of G of degree k is contained in some A-special subgroup of G of degree $k-1$.*
(2) *Let K be an A-invariant subgroup of G and let $K_1, \dots, K_t$ be all the subgroups of the form $K \cap H$, where H is some A-special subgroup of G of degree k. Let $L_1, \dots, L_u$ be all the subgroups of the form $K' \cap J$ where J is some A-special subgroup of G of degree $k+1$. If $K = \langle K_1, \dots, K_t\rangle$, then $K' = \langle L_1, \dots, L_u\rangle$.*
(3) *Let R_k be the subgroup generated by all A-special subgroups of G of degree k. Then $R_k = G^{(k)}$.*
(4) *If $2^k \leqslant r-1$ and H is an A-special subgroup of G of degree k, then H is contained in the kth derived group of $C_G(B)$ for some subgroup $B \leqslant A$ such that $|A/B| \leqslant q^{2^k}$.*
(5) *Suppose that $G = G'$ and let N be an A-invariant subgroup such that $N = [N, G]$. Then for every $k \geqslant 0$ the subgroup N is generated by subgroups of the form $N \cap H$, where H is some A-special subgroup of G of degree k.*
(6) *Let H be an A-special subgroup of G. If N is an A-invariant normal subgroup of G, then the image of H in G/N is an A-special subgroup of G/N.*

Proof. (1) If $k=1$ and H is an A-special subgroup of G of degree 1, then $H = [J_1, J_2] \cap C_G(A_j)$ for a suitable $j \leqslant s$. Observe that $H \leqslant C_G(A_j)$ and the centralizer $C_G(A_j)$ is an A-special subgroup of G of degree 0. Assume that $k \geqslant 2$ and use induction on k. Let H be an A-special subgroup of degree k. We know that there exist A-special subgroups J_1, J_2 of G of degree $k-1$ such that $H = [J_1, J_2] \cap C_G(A_j)$ for suitable $j \leqslant s$. By induction J_i is contained in some A-special subgroup L_i of G of degree $k-2$. Observe that $[L_1, L_2] \cap C_G(A_j)$ is an A-special subgroup of G of degree $k-1$ and $H \leqslant [L_1, L_2] \cap C_G(A_j)$, so the result follows.

(2) Set $M = \langle [K_i, K_j] \mid 1 \leqslant i, j \leqslant t\rangle$. It is clear that each of the subgroups $[K_i, K_j]$ is A-invariant. Thus, by Lemma 2.3 each subgroup $[K_i, K_j]$ is generated by subgroups of the form $[K_i, K_j] \cap C_G(A_l)$, where $l = 1, \dots, s$. Note that each subgroup $[K_i, K_j] \cap C_G(A_l)$ is contained in an A-special subgroup of G of degree $k+1$. Hence M is generated by subgroups of the form $M \cap D$, where D ranges through the set of all A-special subgroups of G of degree $k+1$. If $M^* = M \cap D^*$ is such a subgroup we claim that $[M^*, K_j] \leqslant M$ for every $1 \leqslant j \leqslant t$. Indeed, by (1) we know that there exists some A-special subgroup H of G of degree k such that $D^* \leqslant H$. This implies that M^* is contained in some K_l and so we have $[M^*, K_j] \leqslant [K_l, K_j] \leqslant M$, as desired. Therefore M is normal in K and we conclude that $M = K'$. The result now follows.

(3) If $k = 0$ the result is immediate from Lemma 2.3. Therefore we assume that $k \geqslant 1$ and set $N = R_{k-1}$. By induction on k we assume that $N = G^{(k-1)}$. Let $D_1, D_2, \dots, D_{s_{k-1}}$ be the A-special subgroups of G of degree $k-1$ and $H_1, H_2, \dots, H_{s_k}$ be the A-special subgroups of G of degree k. It follows from (2) that $G^{(k)} = \langle [D_i, D_j] \mid 1 \leqslant i, j \leqslant s_{k-1}\rangle$. Since each subgroup $[D_i, D_j]$ is A-invariant, it follows from Lemma 2.3 that it is generated by subgroups of the form $[D_i, D_j] \cap C_G(A_l)$, where $l = 1, \dots, s$. These are precisely A-special subgroups of G of degree k so the result follows.

(4) If $k = 0$ this is clear because $H = C_G(A_i)$ for a suitable $i \leqslant s$ and $|A/A_i| = q$. Assume that $k \geqslant 1$ and use induction on k. We have $H = [J_1, J_2] \cap C_G(A_j)$ for a suitable $j \leqslant s$ and A-special subgroups J_1, J_2 of G of degree $k-1$. By induction there exist subgroups $B_1, B_2 \leqslant A$ such that $|A/B_i| \leqslant q^{2^{k-1}}$ and $J_i \leqslant C_G(B_i)^{(k-1)}$ where $i = 1, 2$. Set $B = B_1 \cap B_2$. Observe that $H \leqslant [J_1, J_2] \leqslant [C_G(B_1)^{(k-1)}, C_G(B_2)^{(k-1)}] \leqslant [C_G(B)^{(k-1)}, C_G(B)^{(k-1)}]$. Thus $H \leqslant C_G(B)^{(k)}$ and $|A/B| \leqslant q^{2^k}$, as required.

(5) For $k = 0$ this follows from Lemma 2.3. Assume that $k \geqslant 1$ and use induction on k. Let $N_1, \dots, N_t$ be all the subgroups of the form $N \cap H$ where H is some A-special subgroup of degree k and set $M = \langle N_1, \dots, N_t \rangle$. We want to show that $N = M$.

Since $G = G'$ by (3) G can be generated by all A-special subgroups of degree k, for any $k \geqslant 1$. Thus $G = \langle H_1, \dots, H_{s_k} \rangle$, where H_j is A-special subgroup of degree k. Lemma 2.3 shows that for all i and j the commutator $[N_i, H_j]$ is generated by subgroups of the form $[N_i, H_j] \cap C_G(A_l)$, where $l = 1, \dots, s$. Note that each subgroup $[N_i, H_j] \cap C_G(A_l)$ is contained in N since $N = [N, G]$ and on the other hand it is also contained in some A-special subgroup of degree k, so $[N_i, H_j] \cap C_G(A_l) \leqslant N_m$ for a suitable $m \leqslant t$. This implies that M is normal in G.

Let now $L_1, \dots, L_u$ be all the subgroups of the form $N \cap K$ where K is some A-special subgroup of degree $k-1$, so by induction we can assume that $N = \langle L_1, \dots, L_u \rangle$. For all i and j, using the argument as above, it is easy to show that $[L_i, H_j]$ is generated by subgroups of the form $[L_i, H_j] \cap C_G(A_l)$, where $l = 1, \dots, s$ and that each such a subgroup is contained in some N_m, for a suitable $m \leqslant t$.

If $M = 1$, then, for all $m \leqslant t$, $N_m = 1$ and so $[L_i, H_j] = 1$ for all i and j. Hence N is central in G but this is a contradiction because $N = [N, G]$. Assume now that M is a nontrivial subgroup strictly contained in N. Since we have shown that M is normal we can pass to the quotient G/M. In the quotient N/M is central so $[N, G] \leqslant M$ but this contradicts the assumption that $M < N$. Thus we conclude that M must be equal to N.

(6) This is immediate from Lemma 2.2(1) and the definitions. □

4. Some generation results

Throughout this section let q be a prime, G a finite q'-group and A an elementary abelian group of order q^r acting on G. We will show that if P is an A-invariant Sylow p-subgroup of $G^{(d)}$, then it can be generated by its intersections with A-special subgroups of G of degree d.

Theorem 4.1. *Assume $r \geqslant 2$. Let P be an A-invariant Sylow p-subgroup of $G^{(d)}$ for some fixed integer $d \geqslant 0$. Let $P_1, \dots, P_t$ be the subgroups of the form $P \cap H$ where H is some A-special subgroup of G of degree d. Then $P = \langle P_1, \dots, P_t \rangle$.*

We first handle the case where G is a direct product of simple groups.

Lemma 4.2. *Assume that $r \geqslant 2$ and G is a direct product of nonabelian simple groups. Let P be an A-invariant Sylow p-subgroup of G, and for some fixed integer $d \geqslant 0$ let $P_1, \dots, P_t$ be all the subgroups of the form $P \cap H$, where H is some A-special subgroup of G of degree d. Then $P = \langle P_1, \dots, P_t \rangle$.*

Proof. Let $G = S_1 \times \cdots \times S_m$. By induction on the order of G we may assume that A permutes transitively the simple factors $S_1, \dots, S_m$.

We will now use induction on r to show that without loss of generality it can be assumed that A acts on G faithfully. Suppose that some element $a \in A^\#$ acts on G trivially. Thus, $C_G(a) = G$.

Since G is a product of nonabelian simple groups it follows that $[C_G(a), C_G(a)] = G$ and $[C_G(a), C_G(a)] \cap C_G(a) = G$. Thus if $r = 2$, then G itself is an A-special subgroup of degree 1. From this it is easy to see that G is an A-special subgroup of degree d and the lemma follows immediately.

Suppose now that $r \geqslant 3$ and let $A_1, \dots, A_s$ be the maximal subgroups of A. Put $\overline{A} = A/\langle a \rangle$. If $A_1, \dots A_t$ are the maximal subgroups of A containing a, then $\overline{A_1}, \dots, \overline{A_t}$ are maximal subgroups of $\overline{A}$. Then $C_G(\overline{A_i}) = C_G(A_i)$ for all $i \leqslant s$, and so we can consider $\overline{A}$ instead of A and use induction on r.

Thus, from now on we assume that A is faithful on G. Let B be the stabilizer of S_1 in A. Then by Lemma 2.4 B is cyclic. Remark that if $b \notin B$, then $C_G(b)$ is a product of simple groups. Indeed if we consider all the b-orbits, then it is not difficult to see that $C_G(b)$ is the product of the diagonal subgroups of these b-orbits, i.e., $C_G(b)$ is a product of simple groups, one for each b-orbit.

Suppose that $r = 2$ and $B \neq 1$. Let a be a nontrivial element of B and choose $b \in A$ that permutes $S_1, \dots, S_q$. Observe that the case where $A = B$ does not happen because of Lemma 2.4. Since $b \notin B$ from the above remark we know that $C_G(b) = diag(S_1 \times \cdots \times S_q)$ is a diagonal subgroup of G. On the other hand it follows from the Thompson Theorem [18] that $C_{S_1}(a) \neq 1$ and this holds also for the other factors $S_2, \dots, S_q$ because a normalizes each of the simple factors. Thus $C_G(a) = C_{S_1}(a) \times \cdots \times C_{S_q}(a)$ and we have

$$\bigl[C_G(a), C_G(b)\bigr] = \bigl[C_{S_1}(a), diag(S_1 \times \cdots \times S_q)\bigr] \times \cdots \times \bigl[C_{S_q}(a), diag(S_1 \times \cdots \times S_q)\bigr]. \quad (4.1)$$

Furthermore observe that for any $j = 1, \dots, q$

$$\bigl[C_{S_j}(a), diag(S_1 \times \cdots \times S_q)\bigr] = \bigl[C_{S_j}(a), S_j\bigr] = S_j, \quad (4.2)$$

where the first equality follows from the fact that the simple factors commute each other and the second one holds since $[C_{S_j}(a), S_j]$ is a nontrivial normal subgroup of S_j. By (4.1) and (4.2) we see that $[C_G(a), C_G(b)] = S_1 \times \cdots \times S_q = G$. Thus, for any $c \in A^\#$, $C_G(c) = [C_G(a), C_G(b)] \cap C_G(c)$ and so the centralizer $C_G(c)$ is also an A-special subgroup of degree 1. We deduce that, for any $a \in A^\#$, the centralizer $C_G(a)$ is an A-special subgroup of G of any degree. Since Lemma 2.3 tells us that P can be generated by subgroups of the form $P \cap C_G(A_i)$ where A_i are the maximal subgroups of A, the result follows.

Next, assume that $r = 2$ and $B = 1$. Note that A permutes the factors $S_1, \dots, S_{q^2}$ and, for any $a \in A^\#$, the centralizer $C_G(a)$ is a product of q simple groups, one for each a-orbit. Thus $C_G(a)$ is perfect and, in particular, it is an A-special subgroup of any degree for all $a \in A^\#$. The lemma follows.

Finally assume that $r \geqslant 3$. Since B is cyclic we have $|A : B| \geqslant q^2$. Hence we can choose a subgroup E of type (q, q) that intersects B trivially. Note that for all $a \in E^\#$, the centralizer $C_G(a)$ is a product of simple groups. Moreover Lemma 2.3 shows that $P = \prod_{a \in E^\#} C_P(a)$. Therefore it is sufficient to prove that for each $a \in E^\#$ the subgroup $C_G(a) \cap P$ is generated by its intersections with all the A-special subgroups of G of degree d.

Fix a in $E^\#$. Let D be the group of automorphisms induced on $C_G(a)$ by A. By induction $C_G(a) \cap P$ is generated by subgroups of the form $(C_G(a) \cap P) \cap H$, where H ranges through the set of D-special subgroups of $C_G(a)$ of degree d. We now remark that any D-special subgroup of $C_G(a)$ of any degree is in fact an A-special subgroup of G of the same degree. This follows form Definition 3.1 and from the fact that if A_i is a maximal subgroup of A containing a, then there exists a maximal subgroup D_j of D such that $C_{C_G(a)}(D_j) = C_G(A_i)$. Thus we can conclude that $C_G(a) \cap P$ is generated by subgroups of the form $(C_G(a) \cap P) \cap H$, where now H can be regarded as an A-special subgroup of G of degree d. The proof is now complete. □

We now are ready to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Let G be a counterexample of minimal order and let N be a minimal normal A-invariant subgroup of G. Set $X = \langle P_1, \dots, P_t \rangle$. By minimality and Proposition 3.2(6) $PN = XN$. To prove that $P = X$ it is sufficient to show that $P \cap N \leqslant X$.

First suppose that N is a p'-group. In this case the intersection $P \cap N$ is trivial and there is nothing to prove.

Next suppose that N is perfect. Since N is characteristically simple, N is a product of nonabelian simple groups. It follows from Lemma 4.2 that $P \cap N$ is contained in X and we are done.

Thus, it remains to consider the case where N is a p-group. Suppose that $G \neq G'$. By induction we know that every A-invariant Sylow p-subgroup of $G^{(d+1)}$ is generated by its intersections with all the A-special subgroups of G' of degree d. Therefore we can pass to the quotient $G/G^{(d+1)}$ and assume that $G^{(d+1)} = 1$. This implies that $G^{(d)}$ is abelian and so we may assume that $G^{(d)}$ is a p-group. Then $G^{(d)} = P$. It follows from Proposition 3.2(3) that P is generated by A-special subgroups of G of degree d and the result holds.

We are reduced to the case that $G = G'$. Since N is minimal, either $N = [N, G]$ or $N \leqslant Z(G)$.

If $N = [N, G]$ we note that $P \cap N = N$ because N is contained in P. Since $N = [N, G]$, Proposition 3.2(5) shows that N is generated by its intersections with all the A-special subgroups of G of degree d and so $N \leqslant X$, as desired.

Now suppose N central in G. Then N is of order p and either N is contained in every maximal subgroup of P or there exists a maximal subgroup S in P such that $P = NS$.

In the former case $N \leqslant \Phi(P)$. Since we know that $P = XN$, it follows that $P = X$, as required.

In the latter case, by Theorem 2.5, N is also complemented in G and so $G = NH$ for some subgroup $H \leqslant G$. Since N is central, we have $G = N \times H$. This yields a contradiction because we have assumed that $G = G'$. □

We note some consequences of Theorem 4.1. These facts will be useful later on.

Lemma 4.3. *Under the hypothesis of Theorem* 4.1 *let $P^{(l)}$ be the lth derived group of P. Then $P^{(l)} = \langle P^{(l)} \cap P_j \mid 1 \leqslant j \leqslant t\rangle$.*

Proof. First we want to establish the following fact:

$$\begin{aligned}&\text{The group } P^{(l)} \text{ is generated by all the subgroups of the form } P^{(l)} \cap D, \text{ where}\\ &D \text{ ranges through the set of all } A\text{-special subgroups of } G \text{ of degree } d+l.\end{aligned} \tag{4.3}$$

Indeed if $l = 0$ then (4.3) is exactly Theorem 4.1. Assume that $l \geqslant 1$ and use induction on l. Let $L_1, \ldots, L_u$ be all the subgroups of the form $P^{(l)} \cap J$, where J is some A-special subgroup of G of degree $d + l$. By induction $P^{(l-1)}$ is generated by subgroups of the form $P^{(l-1)} \cap D$, where D is some A-special subgroup of G of degree $d + (l - 1)$. It now follows from Proposition 3.2(2) that $P^{(l)} = \langle L_1, \ldots, L_u\rangle$ and this concludes the proof of (4.3).

Now for $l = 0$ the lemma is obvious since by Theorem 4.1 $P = \langle P_1, \ldots, P_t\rangle$, where each subgroup P_j is of the form $P \cap H$ for some A-special subgroup H of G of degree d. Assume that $l \geqslant 1$. Proposition 3.2(1) tells us that every A-special subgroup D of degree $d + l$ is contained in some A-special subgroup H of degree d. Combined with (4.3) this implies that each subgroup of the form $P^{(l)} \cap D$ is contained in P_j, for a suitable $j \leqslant t$. Thus $P^{(l)} = \langle P^{(l)} \cap P_j \mid 1 \leqslant j \leqslant t\rangle$, as required. □

Combining Theorem 4.1 with Lemma 2.1 we obtain a further refinement of Theorem 4.1.

Corollary 4.4. $P = P_1 P_2 \cdots P_t$.

Proof. By Theorem 4.1 we have $P = \langle P_1, \ldots, P_t\rangle$. Since P is nilpotent, in view of Lemma 2.1 it is sufficient to show that

$$\gamma_i(P) = \bigl\langle \gamma_i(P) \cap P_j \bigm| 1 \leqslant j \leqslant t\bigr\rangle, \tag{4.4}$$

for all $i \geqslant 1$.

For $i = 1$ the equality (4.4) is Theorem 4.1. Assume that $i \geqslant 2$. Set $N_j = \gamma_i(P) \cap P_j$ for $j = 1, \ldots, t$ and $N = \langle N_j \mid 1 \leqslant j \leqslant t\rangle$. By Lemma 2.3 $[N_j, P_k]$ can be generated by subgroups of the form $[N_j, P_k] \cap C_G(A_l)$, where $l = 1, \ldots, s$ and each of them is contained in some N_u for suitable $u \leqslant t$. Indeed, on the one hand $[N_j, P_k] \cap C_G(A_l)$ is obviously contained in $\gamma_i(P)$. On the other hand it follows from Proposition 3.2(1) that $[N_j, P_k] \cap C_G(A_l)$ is contained in some A-special subgroup H of degree d. Hence

$$[N_j, P_k] \cap C_G(A_l) \leqslant \gamma_i(P) \cap (P \cap H) = \gamma_i(P) \cap P_u$$

for some $u \leqslant t$. So $[N_j, P_k] \cap C_G(A_l)$ is contained in some N_u as desired. This implies that $[N_j, P_k] \leqslant N$ for all j and k. Therefore N is normal in P.

We can now consider the quotient P/N and observe that for $j = 1, \ldots, t$ the image of the subgroup $\gamma_i(P) \cap P_j$ is trivial. Therefore $\gamma_i(P) \leqslant N$. Since the subgroup N is obviously contained in $\gamma_i(P)$ we conclude that $N = \gamma_i(P)$ and we have (4.4). □

We will also require the following result that is a little stronger than Corollary 4.4.

Corollary 4.5. *For all $l \geqslant 1$ the lth derived group $P^{(l)}$ is the product of the subgroups of the form $P^{(l)} \cap P_j$, where $j = 1, \dots, t$.*

Proof. Recall that by Lemma 4.3 we have

$$P^{(l)} = \langle P^{(l)} \cap P_j \mid 1 \leqslant j \leqslant t \rangle$$

for all $l \geqslant 1$. By using the same argument as in the proof of Corollary 4.4 the result follows. □

5. Useful Lie-theoretic machinery

Let L be a Lie algebra over a field $\mathfrak{k}$. Let k be a positive integer and let $x_1, x_2, \dots, x_k$ be elements of L. We define inductively

$$[x_1] = x_1; \qquad [x_1, x_2, \dots, x_k] = \big[[x_1, x_2, \dots, x_{k-1}], x_k\big].$$

An element $a \in L$ is called ad-nilpotent if there exists a positive integer n such that

$$[x, \underbrace{a, \dots, a}_{n}] = 0 \quad \text{for all } x \in L.$$

If n is the least integer with the above property then we say that a is ad-nilpotent of index n. Let $X \subseteq L$ be any subset of L. By a commutator in elements of X we mean any element of L that can be obtained as a Lie product of elements of X with some system of brackets.

Denote by F the free Lie algebra over $\mathfrak{k}$ on countably many free generators $x_1, x_2, \dots$. Let $f = f(x_1, x_2, \dots, x_n)$ be a non-zero element of F. The algebra L is said to satisfy the identity $f \equiv 0$ if $f(a_1, a_2, \dots, a_n) = 0$ for any $a_1, a_2, \dots, a_n \in L$. In this case we say that L satisfies a polynomial identity, in short, is PI. A deep result of Zelmanov [23], which has numerous important applications to group theory (in particular see [15] for examples where the theorem is used), says that if a Lie algebra L is PI and is generated by finitely many elements all commutators in which are ad-nilpotent, then L is nilpotent. From Zelmanov's result the following theorem can be deduced [10].

Theorem 5.1. *Let L be a Lie algebra over a field $\mathfrak{k}$ generated by $a_1, a_2, \dots, a_m$. Assume that L satisfies an identity $f \equiv 0$ and that each commutator in the generators $a_1, a_2, \dots, a_m$ is ad-nilpotent of index at most n. Then L is nilpotent of $\{f, n, m, \mathfrak{k}\}$-bounded class.*

The next theorem provides an important criterion for a Lie algebra to be PI. It was proved by Bakhturin and Zaicev for soluble groups A [1] and later extended by Linchenko to the general case [12].

Theorem 5.2. *Assume that a finite group A acts on a Lie algebra L by automorphisms in such a manner that $C_L(A)$, the subalgebra formed by fixed elements, is PI. Assume further that the characteristic of the ground field of L is either 0 or prime to the order of A. Then L is PI.*

We will need a corollary of the previous result.

Corollary 5.3. *(See [16].) Let F be the free Lie algebra of countable rank over $\mathfrak{k}$. Denote by F^* the set of non-zero elements of F. For any finite group A there exists a mapping*

$$\phi : F^* \to F^*$$

such that if L and A are as in Theorem 5.2*, and if $C_L(A)$ satisfies an identity $f \equiv 0$, then L satisfies the identity $\phi(f) \equiv 0$.*

Now we turn to groups and for the rest of this section p will denote a fixed prime number. Let G be any group. A series of subgroups

$$(*) \quad G = G_1 \geqslant G_2 \geqslant \cdots$$

is called an N_p-series if $[G_i, G_j] \leqslant G_{i+j}$ and $G_i^p \leqslant G_{pi}$ for all i, j. With any N_p-series $(*)$ of G one can associate a Lie algebra $L^*(G) = \bigoplus L_i^*$ over the field with p elements $\mathbb{F}_p$, where we view each $L_i^* = G_i/G_{i+1}$ as a linear space over $\mathbb{F}_p$. If $x \in G$, let $i = i(x)$ be the largest integer such that $x \in G_i$. We denote by x^* the element xG_{i+1} of $L^*(G)$. The following lemma tells us something about the relationship between the group G and the associated Lie algebra $L^*(G)$.

Lemma 5.4. *(See Lazard [11].) For any $x \in G$ we have $(\mathrm{ad}\, x^*)^p = \mathrm{ad}(x^p)^*$. Consequently, if x is of finite order p^t, then x^* is ad-nilpotent of index at most p^t.*

Let $w = w(x_1, x_2, \ldots, x_n)$ be nontrivial group-word, i.e., a nontrivial element of the free group on free generators $x_1, x_2, \ldots, x_n$. We say that G satisfies the identity $w \equiv 1$ if $w(g_1, \ldots, g_n) = 1$ for any $g_1, g_2, \ldots, g_n \in G$. The next proposition follows from the proof of Theorem 1 in the paper of Wilson and Zelmanov [22].

Proposition 5.5. *Let G be a group satisfying an identity $w \equiv 1$. Then there exists a non-zero multilinear Lie polynomial f over $\mathbb{F}_p$, depending only on p and w, such that for any N_p-series $(*)$ of G the corresponding algebra $L^*(G)$ satisfies the identity $f \equiv 0$.*

In general a group G has many N_p-series; one of the most important is the so-called Jennings–Lazard–Zassenhaus series that can be defined as follows.

Let $\gamma_j(G)$ denote the jth term of the lower central series of G. Set $D_i = D_i(G) = \prod_{jp^k \geqslant i} \gamma_j(G)^{p^k}$. The subgroup D_i is also known as the ith-dimension subgroup of G in characteristic p. These subgroups form an N_p-series of G known as the Jennings–Lazard–Zassenhaus series. Let $L_i = D_i/D_{i+1}$ and $L(G) = \bigoplus L_i$. Then $L(G)$ is a Lie algebra over the field $\mathbb{F}_p$ (see [2, Chapter 11] for more detail). The subalgebra of $L(G)$ generated by $L_1 = D_1/D_2$ will be denoted by $L_p(G)$. The next lemma is a "finite" version of Lazard's criterion for a pro-p group to be p-adic analytic. The proof can be found in [10].

Lemma 5.6. *Suppose that P is a d-generator finite p-group such that the Lie algebra $L_p(P)$ is nilpotent of class c. Then P has a powerful characteristic subgroup of $\{p, c, d\}$-bounded index.*

Remind that powerful p-groups were introduced by Lubotzky and Mann in [13]. A finite p-group G is said to be powerful if and only if $[G, G] \leqslant G^p$ for $p \neq 2$ (or $[G, G] \leqslant G^4$ for $p = 2$). These groups have some nice properties. In particular we will use the following property: if G is a powerful p-group generated by elements of order $e = p^k$, then the exponent of G is e.

Every subspace (or just an element) of $L(G)$ that is contained in D_i/D_{i+1} for some i will be called homogeneous. Given a subgroup H of the group G, we denote by $L(G, H)$ the linear span in $L(G)$ of all homogeneous elements of the form hD_{i+1}, where $h \in D_i \cap H$. Clearly, $L(G, H)$ is always a subalgebra of $L(G)$. Moreover, it is isomorphic with the Lie algebra associated with H using the N_p-series of H formed by $H_i = D_i \cap H$. We also set $L_p(G, H) = L_p(G) \cap L(G, H)$. The proof of the following lemma can be found in [4].

Lemma 5.7. *Suppose that any Lie commutator in homogeneous elements $x_1,\dots,x_r$ of $L(G)$ is ad-nilpotent of index at most t. Let $K=\langle x_1,\dots,x_r\rangle$ and assume that $K\leqslant L(G,H)$ for some subgroup H of G satisfying a group identity $w\equiv 1$. Then there exists some $\{r,t,w,p\}$-bounded number u such that*:

$$\bigl[L(G),\underbrace{K,\dots,K}_{u}\bigr]=0.$$

Lemma 2.2(1) has important implications in the context of associated Lie algebras and their automorphisms. Let G be a group with a coprime automorphism a. Obviously a induces an automorphism of every quotient D_i/D_{i+1}. This action extends to the direct sum $\bigoplus D_i/D_{i+1}$. Thus, a can be viewed as an automorphism of $L(G)$ (or of $L_p(G)$). Set $C_i=D_i\cap C_G(a)$. Then Lemma 2.2(1) shows that

$$C_{L(G)}(a)=\bigoplus C_iD_{i+1}/D_{i+1}, \tag{5.1}$$

and that

$$C_{L_p(G)}(a)=L_p\bigl(G,C_G(a)\bigr). \tag{5.2}$$

This implies that the properties of $C_{L(G)}(a)$ are very much related to those of $C_G(a)$. In particular, Proposition 5.5 shows that if $C_G(a)$ satisfies a certain identity, then $C_{L(G)}(a)$ is PI.

6. Proof of the main result

Our goal in this section is to prove that part (2) of Conjecture 1.3 is correct. More precisely we have the following result.

Theorem 6.1. *Let m be a positive integer, q a prime, and A an elementary abelian group of order q^r, with $r\geqslant 2$. Suppose that A acts as a coprime group of automorphisms on a finite group G. If, for some integer d such that $2^d\leqslant r-1$, the dth derived group of $C_G(a)$ has exponent dividing m for any $a\in A^\#$, then the dth derived group $G^{(d)}$ has $\{m,q,r\}$-bounded exponent.*

First we will consider the particular case where G is a powerful p-group.

Lemma 6.2. *Theorem* 6.1 *is valid if G is powerful.*

Proof. It follows from [2, Exercise 2.1] that $G^{(d)}$ is also powerful. Furthermore, by Proposition 3.2(3), $G^{(d)}$ is generated by A-special subgroups of G of degree d. Since $2^d\leqslant r-1$, Proposition 3.2(4) shows that any A-special subgroup H of G of degree d is contained in $C_G(B)^{(d)}$ for some nontrivial subgroup $B\leqslant A$ and so H is also contained in $C_G(a)^{(d)}$ for some $a\in A^\#$. This implies that $G^{(d)}$ is generated by elements of order dividing m, and so it follows from [2, Lemma 2.5] that the exponent of $G^{(d)}$ divides m. □

We will now handle the case of an arbitrary p-group. The Lie-theoretic techniques that we have described in Section 5 will play a fundamental role in the subsequent arguments.

Lemma 6.3. *Theorem* 6.1 *is valid if G is a p-group.*

Proof. Assume that G is a p-group. By Corollary 4.4 we have

$$G^{(d)}=G_1G_2\cdots G_t, \tag{6.1}$$

where each G_j is an A-special subgroup of G of degree d. It is clear that the number t is $\{q,r\}$-bounded.

Let x be any element of $G^{(d)}$. In view of (6.1) we can write $x = x_1x_2\cdots x_t$, where each x_j belongs to G_j. Since $2^d \leqslant r-1$, by Proposition 3.2(4) each G_j is contained in $C_G(B)^{(d)}$ for some subgroup $B \leqslant A$ such that $|A/B| \leqslant q^{2^d}$. Thus each x_j is contained in some $C_G(a)^{(d)}$ for a suitable $a \in A^{\#}$.

Let Y be the subgroup of G generated by the orbits x_j^A for $j = 1,\dots,t$. Each orbit contains at most q^{r-1} elements so it follows that Y has at most $q^{r-1}t$ generators, each of order dividing m. Since $x \in Y$ and we wish to bound the order of x, it is enough to show that the exponent of Y is $\{m,q,r\}$-bounded.

Set $Y_j = G_j \cap Y$ for $j = 1,\dots,t$ and note that every $Y_j \leqslant C_G(a)^{(d)}$ for a suitable $a \in A^{\#}$. Since $Y = \langle x_1^A,\dots,x_t^A\rangle$ and every G_j is an A-invariant subgroup we have $Y = \langle Y_1,\dots,Y_t\rangle$. By applying Lemma 2.1 we see that $Y = Y_1Y_2\cdots Y_t$.

Let $L = L_p(Y)$ and let $V_1,\dots,V_t$ be the images of $Y_1,\dots,Y_t$ in $Y/\Phi(Y)$. It follows that the Lie algebra L is generated by $V_1,\dots,V_t$.

Let W be a subspace of L. We say that W is a *special subspace* of weight 1 of L if and only if $W = V_j$ for some $j \leqslant t$ and say that W is a special subspace of weight $\varphi \geqslant 2$ if $W = [W_1, W_2] \cap C_L(A_k)$, where W_1, W_2 are some special subspaces of L of weight φ_1 and φ_2 such that $\varphi_1 + \varphi_2 = \varphi$ and A_k is some maximal subgroup of A for a suitable k.

We wish to show that every special subspace W of L corresponds to a subgroup of an A-special subgroup of G of degree d. We argue by induction on the weight φ. If $\varphi = 1$, then $W = V_j$ and so W corresponds to Y_j for some $j \leqslant t$. Assume that $\varphi \geqslant 2$ and write $W = [W_1, W_2] \cap C_L(A_k)$. By induction we know that W_1, W_2 correspond respectively to some J_1, J_2 which are subgroups of some A-special subgroups of G degree d. Note that $[W_1, W_2]$ is contained in the image of $[J_1, J_2]$. This implies that the special subspace W corresponds to a subgroup of $[J_1, J_2] \cap C_G(A_k)$ which, by Proposition 3.2(1), is contained in some A-special subgroup of G of degree d, as desired. Moreover it follows from Proposition 3.2(4) that every element of W corresponds to some element of $C_G(a)^{(d)}$ for some $a \in A^{\#}$ and so, by Lemma 5.4, it is ad-nilpotent of index at most m.

From the previous argument we deduce that $L = \langle V_1,\dots,V_t\rangle$ is generated by ad-nilpotent elements of index at most m but we cannot claim that every Lie commutator in these generators is again in some special subspace of L and hence it is ad-nilpotent of bounded index. To overcome this difficulty we extend the ground field $\mathbb{F}_p$ by a primitive qth root of unity ω and put $\bar{L} = L \otimes \mathbb{F}_p[\omega]$. We view $\bar{L}$ as a Lie algebra over $\mathbb{F}_p[\omega]$ and it is natural to identify L with the $\mathbb{F}_p$-subalgebra $L \otimes 1$ of $\bar{L}$. In what follows we write $\overline{X}$ to denote $X \otimes \mathbb{F}_p[\omega]$ for some subspace X of L. Note that if an element $x \in L$ is ad-nilpotent, then the "same" element $x \otimes 1$ is also ad-nilpotent in $\bar{L}$. We will say that an element of $\bar{L}$ is homogeneous if it belongs to $\bar{S}$ for some homogeneous subspace S of L.

Let W be a special subspace of L. We claim that

$$\text{there exists an } \{m,q\}\text{-bounded number } u \text{ such that every element } w \text{ of } \overline{W} \text{ is ad-nilpotent of index at most } u. \tag{6.2}$$

Since w is a homogeneous element of $\bar{L}$ it can be written as

$$w = l_0 \otimes 1 + l_1 \otimes \omega + \cdots + l_{q-2} \otimes \omega^{q-2},$$

for suitable homogeneous elements $l_0,\dots,l_{q-2}$ of W. The elements $l_0,\dots,l_{q-2}$ correspond to some $x_0,\dots,x_{q-2}$ of Y that belong to some A-special subgroup of degree d and so in particular $x_0,\dots,x_{q-2}$ are elements of $C_G(a)^{(d)}$ for some $a \in A^{\#}$. Set $H = \langle x_0,\dots,x_{q-2}\rangle$ and $K = \langle l_0,\dots,l_{q-2}\rangle$. Since H has exponent m and $K \leqslant L_p(Y,H)$, Lemma 5.7 shows that there exists an $\{m,q\}$-bounded number u such that

$$[L, \underbrace{K,\dots,K}_{u}] = 0. \tag{6.3}$$

Obviously (6.3) implies that

$$[\bar{L}, \underbrace{\overline{K}, \dots, \overline{K}}_{u}] = 0. \tag{6.4}$$

Since w lies in $\overline{K}$, (6.2) follows.

The group A acts naturally on $\bar{L}$ and now the ground field is a splitting field for A. Since Y can be generated by at most $q^{r-1}t$ elements, we can choose elements $v_1, \dots, v_s$ in $\overline{V_1} \cup \dots \cup \overline{V_t}$ with $s \leqslant q^{r-1}t$ that generate the Lie algebra $\bar{L}$ and each of them is a common eigenvector for all transformations from A.

Now let v be any Lie commutator in $v_1, \dots, v_s$. We wish to show that v belongs to some $\overline{W}$, where W is a special subspace of L. If v has weight 1 there is nothing to prove. Assume v has weight at least 2. Write $v = [w_1, w_2]$ for some $w_1 \in \overline{W_1}$ and $w_2 \in \overline{W_2}$, where W_1, W_2 are two special subspaces of L of smaller weights. It is clear that v belongs to $[\overline{W_1}, \overline{W_2}]$. Note that any commutator in common eigenvectors is again a common eigenvector. Therefore v is a common eigenvector and it follows that there exists some maximal subgroup A_l of A such that $v \in C_{\bar{L}}(A_l)$. Thus $v \in [\overline{W_1}, \overline{W_2}] \cap C_{\bar{L}}(A_l)$. Hence v lies in $\overline{W}$, where W is the special subspace of L of the form $[W_1, W_2] \cap C_L(A_l)$ and so by (6.2) v is ad-nilpotent of bounded index. This proves that

$$\text{any commutator in } v_1, \dots, v_s \text{ is ad-nilpotent of index at most } u. \tag{6.5}$$

Remind that $C_L(a) = L_p(Y, C_Y(a))$. Proposition 5.5 shows that $C_L(a)$ satisfies a multilinear polynomial identity of $\{m, q\}$-bounded degree. This also holds in $C_{\bar{L}}(a) = \overline{C_L(a)}$. Therefore Corollary 5.3 implies that $\bar{L}$ satisfies a polynomial identity of $\{m, q\}$-bounded degree. Combining this with (6.2) and (6.5) we are now able to apply Theorem 5.1. Thus $\bar{L}$ is nilpotent of $\{m, q, r\}$-bounded class and the same holds for L.

Since Y is a p-group and $L = L_p(Y)$ is nilpotent of bounded class, it follows from Lemma 5.6 that Y has a characteristic powerful subgroup K of $\{m, q, r\}$-bounded index. By Lemma 6.2 $K^{(d)}$ has bounded exponent and so we can pass to the quotient $Y/K^{(d)}$ and assume that Y is of $\{m, q, r\}$-bounded derived length. We now recall that $Y = Y_1 Y_2 \dots Y_t$ and each Y_j is contained is some G_j. From the results obtained in Section 4 also each derived group $Y^{(i)}$ is a product of subgroups of the form $Y^{(i)} \cap Y_j$. Thus every $Y^{(i)}$ can be generated by elements whose orders divide m. Since the derived length of Y is bounded, we conclude that Y has $\{m, q, r\}$-bounded exponent, as required. □

Finally we are ready to complete the proof of Theorem 6.1.

Proof of Theorem 6.1. Note that it suffices to prove that there is a bound, depending only on m, q and r, on the exponent of a Sylow p-subgroup of $G^{(d)}$ for each prime p.

Indeed, let $\pi(G^{(d)})$ be the set of prime divisors of $|G^{(d)}|$. Choose $p \in \pi(G^{(d)})$. It follows from Lemma 2.2(2) that $G^{(d)}$ possesses an A-invariant Sylow p-subgroup, say P. By Corollary 4.4, $P = P_1 P_2 \cdots P_t$, where each P_j is of the form $P \cap H$ for some A-special subgroup H of G of degree d. Combining this fact with Proposition 3.2(4) we see that each P_j is contained in $C_G(B)^{(d)}$ for a suitable subgroup B of A and thus $P_j \leqslant C_G(a)^{(d)}$, for some $a \in A^{\#}$. Since the exponent of $C_G(a)^{(d)}$ divides m, so does p.

From Lemma 6.3 we know that $P^{(d)}$ has $\{m, q, r\}$-bounded exponent. Moreover by Lemma 4.3 the subgroup $P^{(d-1)}$ is generated by subgroups of the form $P^{(d-1)} \cap P_j$, for $j = 1, \dots, t$, so in particular $P^{(d-1)}$ is generated by elements of order dividing m. Since $P^{(d)} = (P^{(d-1)})'$ has bounded exponent it is clear that also the exponent of $P^{(d-1)}$ is $\{m, q, r\}$-bounded. Repeating the same argument several times we see that all subgroups $P^{(d-2)}, \dots, P'$ and P are generated by elements whose orders divide m and so we conclude that P has $\{m, q, r\}$-bounded exponent, as desired. This completes the proof. □

7. The other part of the conjecture

In this last section we will deal with part (1) of Conjecture 1.3. The proof of that part is similar to that of part (2) but in fact it is easier. Therefore we will not give a detailed proof here but rather describe only steps where the proof of part (1) is somewhat different from that of part (2).

The definition of A-special subgroups of G needs to be modified in the following way.

Definition 7.1. Let A be an elementary abelian q-group acting on a finite q'-group G. Let $A_1, \dots, A_s$ be the subgroups of index q in A and H a subgroup of G.

- We say that H is a γ-A-special subgroup of G of degree 1 if and only if $H = C_G(A_i)$ for suitable $i \leqslant s$.
- Suppose that $k \geqslant 2$ and the γ-A-special subgroups of G of degree $k-1$ are defined. Then H is a γ-A-special subgroup of G of degree k if and only if there exists a γ-A-special subgroup J of G of degree $k-1$ such that $H = [J, C_G(A_i)] \cap C_G(A_j)$ for suitable $i, j \leqslant s$.

The next proposition is similar to Proposition 3.2.

Proposition 7.2. *Let A be an elementary abelian q-group of order q^r with $r \geqslant 2$ acting on a finite q'-group G and $A_1, \dots, A_s$ the maximal subgroups of A. Let $k \geqslant 1$ be an integer.*

(1) *If $k \geqslant 2$, then every γ-A-special subgroup of G of degree k is contained in some γ-A-special subgroup of G of degree $k-1$.*
(2) *Let R_k be the subgroup generated by all γ-A-special subgroups of G of degree k. Then $R_k = \gamma_k(G)$.*
(3) *If $k \leqslant r-1$ and H is a γ-A-special subgroup of G of degree k, then $H \leqslant \gamma_k(C_G(B))$ for some subgroup $B \leqslant A$ such that $|A/B| \leqslant q^k$.*
(4) *Suppose that $G = G'$ and let N be an A-invariant subgroup such that $N = [N, G]$. Then for every $k \geqslant 1$ the subgroup N is generated by subgroups of the form $N \cap H$, where H is some γ-A-special subgroup of G of degree k.*
(5) *Let H be a γ-A-special subgroup of G. If N is an A-invariant normal subgroup of G, then the image of H in G/N is a γ-A-special subgroup of G/N.*

The above properties of γ-A-special subgroups are essential in the proof of the following generation result, which is analogous to Theorem 4.1.

Theorem 7.3. *Assume $r \geqslant 2$. Let P be an A-invariant Sylow p-subgroup of $\gamma_{r-1}(G)$. Let $P_1, \dots, P_t$ be all the subgroups of the form $P \cap H$ where H is some γ-A-special subgroup of G of degree $r-1$. Then $P = \langle P_1, \dots, P_t \rangle$.*

From this one can deduce

Theorem 7.4. *Let m be a positive integer, q a prime and A an elementary abelian group of order q^r with $r \geqslant 2$ acting on a finite q'-group G. If $\gamma_{r-1}(C_G(a))$ has exponent dividing m for any $a \in A^{\#}$, then $\gamma_{r-1}(G)$ has $\{m, q, r\}$-bounded exponent.*

The above theorem shows that part (1) of Conjecture 1.3 is correct. The proof of Theorem 7.4 can be obtained in the same way as that of Theorem 6.1 with only obvious changes required. Thus, we omit the further details.

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