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On finite groups in which coprime commutators are covered by few cyclic subgroups

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A B S T R A C T

The coprime commutators γ_j^* and δ_j^* were recently introduced as a tool to study properties of finite groups that can be expressed in terms of commutators of elements of coprime orders. Every element of a finite group G is both a γ_1^*-commutator and a δ_0^*-commutator. Now let $j \geqslant 2$ and let X be the set of all elements of G that are powers of γ_{j-1}^*-commutators. An element g is a γ_j^*-commutator if there exist $a \in X$ and $b \in G$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. For $j \geqslant 1$ let Y be the set of all elements of G that are powers of δ_{j-1}^*-commutators. An element g is a δ_j^*-commutator if there exist $a, b \in Y$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. The subgroups of G generated by all γ_j^*-commutators and all δ_j^*-commutators are denoted by $\gamma_j^*(G)$ and $\delta_j^*(G)$, respectively. For every $j \geqslant 2$ the subgroup $\gamma_j^*(G)$ is precisely the last term $\gamma_\infty(G)$ of the lower central series of G, while for every $j \geqslant 1$ the subgroup $\delta_j^*(G)$ is precisely the last term of the lower central series of $\delta_{j-1}^*(G)$, that is, $\delta_j^*(G) = \gamma_\infty(\delta_{j-1}^*(G))$.

In the present paper we prove that if G possesses m cyclic subgroups whose union contains all γ_j^*-commutators of G, then $\gamma_j^*(G)$ contains a subgroup Δ, of m-bounded order, which is normal in G and has the property that $\gamma_j^*(G)/\Delta$ is cyclic.

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If $j \geqslant 2$ and G possesses m cyclic subgroups whose union contains all δ_j^*-commutators of G, then the order of $\delta_j^*(G)$ is m-bounded.

1. Introduction

A covering of a group G is a family $\{S_i\}_{i\in I}$ of subsets of G such that $G = \bigcup_{i\in I} S_i$. If $\{H_i\}_{i\in I}$ is a covering of G by subgroups, it is natural to ask what information about G can be deduced from properties of the subgroups H_i. In the case where the covering is finite actually quite a lot about the structure of G can be said. In particular, as was first pointed out by Baer (see [10, p. 105]), a group covered by finitely many cyclic subgroups is either cyclic or finite. More recently Fernández-Alcober and Shumyatsky proved that if G is a group in which the set of all commutators is covered by finitely many cyclic subgroups, then G' is either finite or cyclic [4]. This suggests the question about the structure of a group in which the set of all γ_j-commutators (or of all δ_j-commutators) is covered by finitely many cyclic subgroups. Here the words γ_j and δ_j are defined by the positions $\gamma_1 = \delta_0 = x_1$, $\gamma_{j+1} = [\gamma_j, x_{j+1}]$ and $\delta_{j+1} = [\delta_j, \delta_j]$.

In [3] Cutolo and Nicotera showed that if G is a group in which the set of all γ_j-commutators is covered by finitely many cyclic subgroups, then $\gamma_j(G)$ is finite-by-cyclic. They also showed that $\gamma_j(G)$ can be neither cyclic nor finite. It is still unknown whether a similar result holds for the derived words δ_j.

In [11] the coprime commutators γ_j^* and δ_j^* were introduced as a tool to study properties of finite groups that can be expressed in terms of commutators of elements of coprime orders. For the reader's convenience we recall here the definitions. Let G be a finite group. Every element of G is both a γ_1^*-commutator and a δ_0^*-commutator. Now let $j \geqslant 2$ and let X be the set of all elements of G that are powers of γ_{j-1}^*-commutators. An element g is a γ_j^*-commutator if there exist $a \in X$ and $b \in G$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. For $j \geqslant 1$ let Y be the set of all elements of G that are powers of δ_{j-1}^*-commutators. The element g is a δ_j^*-commutator if there exist $a, b \in Y$ such that $g = [a, b]$ and $(|a|, |b|) = 1$. The subgroups of G generated by all γ_j^*-commutators and all δ_j^*-commutators will be denoted by $\gamma_j^*(G)$ and $\delta_j^*(G)$, respectively. One can easily see that if N is a normal subgroup of G and x an element whose image in G/N is a γ_j^*-commutator (respectively a δ_j^*-commutator), then there exists a γ_j^*-commutator y in G (respectively a δ_j^*-commutator) such that $x \in yN$.

It was shown in [11] that $\gamma_j^*(G) = 1$ if and only if G is nilpotent and $\delta_j^*(G) = 1$ if and only if the Fitting height of G is at most j. It follows that for every $j \geqslant 2$ the subgroup $\gamma_j^*(G)$ is precisely the last term of the lower central series of G (which throughout the paper will be denoted by $\gamma_\infty(G)$) while for every $j \geqslant 1$ the subgroup $\delta_j^*(G)$ is precisely the last term of the lower central series of $\delta_{j-1}^*(G)$, that is, $\delta_j^*(G) = \gamma_\infty(\delta_{j-1}^*(G))$.

In the present paper we prove the following theorem.

Theorem 1.1. *Let j be a positive integer and G a finite group that possesses m cyclic subgroups whose union contains all γ_j^*-commutators of G. Then $\gamma_j^*(G)$ contains a subgroup Δ, of m-bounded order, which is normal in G and has the property that $\gamma_j^*(G)/\Delta$ is cyclic.*

We note that the above result seems to be new even in the case where $j = 1$. Thus, one immediate corollary of Theorem 1.1 is that a finite group covered by m cyclic subgroups has a normal subgroup Δ of m-bounded order with the property that G/Δ is cyclic. This can be easily extended to arbitrary groups.

Corollary 1.2. *Let G be a (possibly infinite) group covered by m cyclic subgroups. Then G has a finite normal subgroup Δ, of m-bounded order, such that G/Δ is cyclic.*

Indeed, let G be as in the above corollary. The classical result of B.H. Neumann [8] tells us that G has a cyclic subgroup of finite index. Therefore G is residually finite and all finite quotients of G satisfy the hypothesis of Theorem 1.1. Hence, G has a normal subgroup Δ of m-bounded order with the property that G/Δ is cyclic.

We also mention that in Theorem 1.1 the subgroup $\gamma_j^*(G)$ is (of bounded order)-by-cyclic and so we observe here a phenomenon related to what was proved by Cutolo and Nicotera for the verbal subgroups $\gamma_j(G)$.

Having dealt with Theorem 1.1, it is natural to look at finite groups in which δ_j^*-commutators can be covered by few cyclic subgroups. Since for $j \leqslant 1$ any δ_j^*-commutator is a γ_{j+1}^*-commutator, the interesting cases occur when $j \geqslant 2$.

Theorem 1.3. *Let $j \geqslant 2$ and G be a finite group that possesses m cyclic subgroups whose union contains all δ_j^*-commutators of G. Then the order of $\delta_j^*(G)$ is m-bounded.*

Throughout the paper we use the expression "$(a, b, \ldots)$-bounded" to mean that the bound is a function of the parameters $a, b, \ldots$. Henceforth all groups considered in this paper will be finite and the term "group" will mean "finite group".

2. Preliminaries

We begin with some results about coprime actions of groups. Let H and K be subgroups of a group G. We denote by $[K, H]$ the subgroup of G generated by $\{[k, h]\colon k \in K, h \in H\}$, and by $[K,_i H]$ the subgroup $[[K,_{i-1} H], H]$ for $i \geqslant 2$. If G is a p-group, we denote by $\Omega_1(G)$ the subgroup of G generated by its elements of order p.

Lemma 2.1. *(See [5], Theorems 5.2.3, 5.2.4 and 5.3.6.) Let A and G be groups with $(|G|, |A|) = 1$ and suppose that A acts on G. Then we have*

(1) $[G, A, A] = [G, A]$;
(2) *If G is an abelian p-group, then $G = C_G(A) \times [G, A]$;*
(3) *If G is an abelian p-group and A acts trivially on $\Omega_1(G)$, then A acts trivially on G.*

Lemma 2.2. *Let G be an abelian p-group and α a coprime automorphism of G. If $[G,\alpha]$ is cyclic, then $[G,\alpha]=[G,\alpha^i]$ for any integer i such that $\alpha^i\neq 1$.*

Proof. By Lemma 2.1(2) we have $G=C_G(\alpha)\times[G,\alpha]$. Suppose that $\alpha^i\neq 1$ and $[G,\alpha]\neq[G,\alpha^i]$. Then $C_{[G,\alpha]}(\alpha^i)\neq 1$. Since $[G,\alpha]$ is cyclic, we conclude that $\Omega_1([G,\alpha])\leqslant C_{[G,\alpha]}(\alpha^i)$ and therefore α^i acts trivially on $[G,\alpha]$. This implies that $\alpha^i=1$, a contradiction. □

Lemma 2.3. *Let G be a cyclic group faithfully acted on by a group A. The following holds.*

(1) *The group A is abelian;*
(2) *If G is a p-group and A is a p'-group, then A is cyclic.*

Proof. Both claims are immediate from the well-known fact that the group of automorphisms of the additive cyclic group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic with the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$. □

Lemma 2.4. *Let $j\geqslant 2$ and G be a group containing a normal subgroup N. If $N\leqslant\delta_j^*(G)$ and $\delta_j^*(G)/N$ is cyclic, then $\delta_j^*(G)=N$.*

Proof. We pass to the quotient G/N and without loss of generality assume that $N=1$. Therefore $\delta_j^*(G)$ is cyclic and so by Lemma 2.3(1) we have $\delta_j^*(G)\leqslant Z(G')$. It follows that $\delta_{j-1}^*(G)$ is nilpotent and, since $\delta_j^*(G)=\gamma_\infty(\delta_{j-1}^*(G))$, we deduce that $\delta_j^*(G)=1$. This completes the proof. □

The following lemma is well-known. The proof can be found for example in [1].

Lemma 2.5. *Let G be a metanilpotent group, P a Sylow p-subgroup of $\gamma_\infty(G)$ and H a Hall p'-subgroup of G. Then $P=[P,H]$.*

The next lemma will be very useful.

Lemma 2.6. *Let $y_1,\dots,y_{j+1}$ be powers of δ_j^*-commutators in G. Suppose that the elements $y_1,\dots,y_{j+1}$ normalize a subgroup N such that $(|y_i|,|N|)=1$ for every $i=1,\dots,j+1$. Then for every $g\in N$ the element $[g,y_1,\dots,y_{j+1}]$ is a δ_{j+1}^*-commutator.*

Proof. We note that all elements of the form $[g,y_1,\dots,y_i]$ are of order prime to $|y_{i+1}|$. An easy induction on i shows that whenever $i\leqslant j$ the element $[g,y_1,\dots,y_{i+1}]$ is a δ_{i+1}^*-commutator. The lemma follows. □

Lemma 2.7. *Let G be a group, P a normal p-subgroup of G and x a p'-element in G. Let $j\geqslant 1$ be an integer. Then we have*

(1) *The subgroup* $[P,x]$ *is generated by* γ_j^**-commutators.*
(2) *If* P *is abelian, then every element of* $[P,x]$ *is a* γ_j^**-commutator.*
(3) *If* x *is a power of a* δ_{j-1}^**-commutator, then* $[P,x]$ *is generated by* δ_j^**-commutators.*
(4) *If* x *is a power of a* δ_{j-1}^**-commutator and* P *is abelian, then every element of* $[P,x]$ *is a* δ_j^**-commutator.*

Proof. In view of Lemma 2.1(1) $[P,x] = [P,\underbrace{x,\dots,x}_{j-1}]$. Suppose first that P is abelian. Note that every element of the form $[g,\underbrace{x,\dots,x}_{j-1}]$, with $g \in P$, is a γ_j^*-commutator. Since P is abelian, every element of $[P,x]$ is of the form $[g,\underbrace{x,\dots,x}_{j-1}]$ for a suitable $g \in P$ and therefore every element of $[P,x]$ is a γ_j^*-commutator. Now drop the assumption that P is abelian. We wish to show that $[P,x]$ is generated by γ_j^*-commutators. Passing to the quotient $G/\Phi(P)$ we may assume that P is elementary abelian and use the result for the abelian case. This proves Claims (1) and (2).

The proof of Claims (3) and (4) follows a similar argument using Lemma 2.6. □

The well-known Focal Subgroup Theorem [5, Theorem 7.3.4] states that if G is a group and P a Sylow p-subgroup of G, then $P \cap G'$ is generated by the set of commutators $\{[g,z] \mid g \in G,\ z \in P,\ [g,z] \in P\}$. In particular, it follows that $P \cap G'$ can be generated by commutators lying in P. This observation led to the question on generation of Sylow subgroups of verbal subgroups of finite groups. The main result of [2] is that $P \cap w(G)$ is generated by powers of w-values, whenever w is a multilinear commutator word. More recently an analogous result on the generation of Sylow subgroups of $\delta_j^*(G)$ in the case where G is soluble was proved in [1]. More precisely we have the following lemma that we will need later on.

Lemma 2.8. *(See [1], Lemma 2.6.) Let* $j \geqslant 0$*. Let* G *be a soluble group and* P *a Sylow* p*-subgroup of* G*. Then* $P \cap \delta_j^*(G)$ *is generated by powers of* δ_j^**-commutators.*

It is natural to conjecture that Lemma 2.8 actually holds for all finite groups. In particular, the corresponding result in [2] was proved without the assumption that G is soluble. It seems though that proving Lemma 2.8 for arbitrary groups is a complicated task. Indeed, one of the tools used in [2] is the proof of the Ore Conjecture by Liebeck, O'Brien, Shalev, and Tiep [7] that every element of any nonabelian finite simple group is a commutator. Recently it was conjectured in [11] that every element of a finite simple group is a commutator of elements of coprime orders. If this is confirmed, proving Lemma 2.8 for arbitrary groups would be easy. However the conjecture that every element of a finite simple group is a commutator of elements of coprime orders is proved only for the alternating groups [11] and the groups $\mathrm{PSL}(2,q)$ [9].

Lemma 2.9. *Let G be a noncyclic p-group that can be covered by m cyclic subgroups. Then $|G|$ is m-bounded.*

Proof. To start with, we consider the case where G is abelian. We notice that the minimal number of generators of G is at most m and therefore it is sufficient to bound the exponent of G. The group G contains an elementary abelian subgroup, say J, of order p^2. One requires precisely $p+1$ cyclic subgroups to cover J. Hence $p+1 \leqslant m$. Let the exponent of G be p^n. Since $p \leqslant m-1$, it is sufficient to bound n. We assume that $n \geqslant 2$. Choose an element $a \in G$ whose order is p^n and an element $b \in G \setminus \langle a \rangle$ of order p. Set $H = \langle a, b \rangle$. It is clear that any covering of H by cyclic subgroups requires some subgroups of order p^n. Further, the element $a^p b$ has order p^{n-1} and it is not contained in any cyclic subgroup of order p^n. Therefore any covering of H by cyclic subgroups requires also some subgroups of order p^{n-1}. Assuming that $n \geqslant 3$ we now consider the element $a^{p^2} b$. This has order p^{n-2} and is not contained in any cyclic subgroup of order p^{n-1}. Thus any covering of H by cyclic subgroups requires some subgroups of order p^{n-2}. It now becomes clear that any covering of H by cyclic subgroups requires some subgroups of all possible orders $p^n, p^{n-1}, \dots, p$. It follows that $n \leqslant m$ and in the case where G is abelian the lemma is proved.

We now drop the assumption that G is abelian. Let N be a maximal normal abelian subgroup. Then $N = C_G(N)$. If N is noncyclic, then by the previous argument $|N|$ is m-bounded and, since G/N embeds in $Aut\,N$, the order of G is m-bounded, too. Hence we assume that N is cyclic of order p^n. The quotient G/G' is abelian and noncyclic. Hence G/G' contains an elementary abelian subgroup of order p^2. We have remarked in the previous paragraph that the existence of such a subgroup implies that $p \leqslant m-1$ and so now it is sufficient to bound n. Let y be an element of least order in $G \setminus N$. In view of [5, Theorem 5.4.4] the order of y is either p or 4. Let $P = N\langle y \rangle$. Since $C_P(y)$ is abelian, the previous paragraph shows that $|C_P(y)|$ is m-bounded. Hence, it is sufficient to bound the index of $C_N(y)$ in N. This is precisely the order of the subgroup $[N, y]$. Observe that all elements in the coset $[N, y]y^{-1}$ are conjugate to y^{-1} and so P contains at least $|[N, y]|$ elements of order $|y|$ (which is either p or 4). Any nontrivial cyclic p-group contains exactly $p-1$ elements of order p and at most two elements of order 4. Therefore one requires at least $|[N, y]|/p$ cyclic subgroups in P to cover the coset $[N, y]y^{-1}$. Hence $|[N, y]|/p \leqslant m$ and since $p \leqslant m-1$, we deduce that $|[N, y]| \leqslant m(m-1)$. The proof is complete. □

We close this preliminary section with the following results about coprime actions.

Lemma 2.10. *Let j be a positive integer, P a p-group of class c and α a p'-automorphism of P. Suppose that P has m cyclic subgroups whose union contains all elements of the form $[x, \underbrace{\alpha, \dots, \alpha}_{j}]$, with $x \in P$. If $[P, \alpha]$ is noncyclic, then the order of $[P, \alpha]$ is (c, m)-bounded.*

Proof. By Lemma 2.1(1) we have $P = [P, \alpha] = [P, \underbrace{\alpha, \ldots, \alpha}_{j}]$. We argue by induction on the nilpotency class c. If $c = 1$, then P is abelian and it consists of elements of the form $[x, \underbrace{\alpha, \ldots, \alpha}_{j}]$. It follows that P can be covered by m cyclic subgroups and by Lemma 2.9 the order of P is m-bounded.

Assume $c \geqslant 2$ and pass to the quotient $\overline{P} = P/P'$. Of course $\overline{P}$ is not cyclic and abelian. Hence by the argument in the previous paragraph the order of $\overline{P}$ is m-bounded and since P is nilpotent of class c, it follows that $|P|$ is (c, m)-bounded, as desired. □

Lemma 2.11. *Let A be a noncyclic p'-group of automorphisms of a noncyclic abelian p-group G. Then there exists $a \in A$ such that $[G, a]$ is noncyclic.*

Proof. Suppose that the lemma is false and $[G, a]$ is cyclic for every a in A.

Firstly we consider the case where A is abelian. Choose a nontrivial element $a_1 \in A$. The cyclic subgroup $[G, a_1]$ is A-invariant and, by Lemma 2.3, the quotient $A/C_A([G, a_1])$ is cyclic. In particular $C_A([G, a_1]) \neq 1$ so we choose a nontrivial element $a_2 \in C_A([G, a_1])$. Since a_2 centralizes $[G, a_1]$, it follows that $[G, a_1][G, a_2]$ is not cyclic. Moreover, it is clear that a_1 centralizes $[G, a_2]$. Hence, $[G, a_1][G, a_2] \leqslant [G, a_1 a_2]$ and this is a contradiction. Thus, in the case where A is abelian the result follows.

Suppose now that A is nilpotent. If A contains a noncyclic abelian subgroup, then the result follows from the previous paragraph. Hence, without loss of generality, we suppose that every abelian subgroup of A is cyclic. It follows (see for example [5, Theorem 4.10(ii), p. 199]) that A is isomorphic to $Q \times C$, where Q is the generalized quaternion group and C is a cyclic group of odd order. By Lemma 2.2 for any a in A and any integer i such that $a^i \neq 1$ we have $[G, a] = [G, a^i]$. Let a_0 be the unique involution of A. It is clear that a_0 is contained in all maximal cyclic subgroups of A. It follows that $[G, a] = [G, a_0]$ for all a in A. Hence we conclude that $[G, A] = [G, a_0]$ which is cyclic. By Lemma 2.1 A acts faithfully on $[G, a_0]$ and, in view of Lemma 2.3(2), the group A must be cyclic. This is a contradiction.

Finally we can drop the assumption that A is nilpotent. If A contains at least one noncyclic nilpotent subgroup, we use the previous case. Thus, we assume that all nilpotent subgroups in A are cyclic and in this case A is soluble. Let $F = F(A)$ be the Fitting subgroup of A. Of course we can assume that A is not nilpotent and so we can choose a subgroup Q of F of prime order q such that Q is not contained in $Z(A)$. Then there exists a q'-element a in A such that $[Q, a] = Q$. The element a acts on $[G, Q]$, which is a cyclic p-group. Thus $Q\langle a \rangle$ acts on $[G, Q]$, but this leads to a contradiction since by Lemma 2.3(1) the group of automorphisms of a cyclic group is abelian. □

3. Theorem 1.3

Turull introduced in [12] the concept of an irreducible B-tower and showed that a soluble group G has Fitting height h if and only if h is maximal such that there exists an irreducible tower of height h consisting of subgroups of G (see Lemmas 1.4 and 1.9(3) in [12]). We will now remind the reader some of the properties of subgroups forming an irreducible tower (we require only the case $B = 1$ and refer to these objects simply as "towers").

Let P_i, where $i = 1, \dots, h$ be subgroups of G forming a tower of height h. Then we have

(1) P_i is a p_i-group (p_i a prime) for $i = 1, \dots, h$.
(2) P_i normalizes P_j for $i < j$.
(3) $p_i \neq p_{i+1}$ for $i = 1, \dots, h-1$.
(4) $[P_i, P_{i-1}] = P_i$ for $i = 2, \dots, h$.
(5) Let $\bar{P}_i = P_i/C_{P_i}(\bar{P}_{i+1})$ for $i = 1, \dots, h-1$ and $\bar{P}_h = P_h$. Then $\phi(\phi(\bar{P}_i)) = 1$, $\phi(\bar{P}_i) \leqslant Z(\bar{P}_i)$. Moreover P_{i-1} centralizes $\phi(\bar{P}_i)$ for $i = 2, \dots, h$. Here ϕ denotes the Frattini subgroup.

In the next few lemmas we will assume that $\delta^*_{j+1}(G) = 1$. Therefore $\delta^*_j(G)$ is nilpotent and so any Sylow subgroups of $\delta^*_j(G)$ is normal in G.

Lemma 3.1. *Let p be a prime, j a positive integer and G a group such that $\delta^*_{j+1}(G) = 1$. Suppose that $\delta^*_j(G)$ is a nontrivial abelian p-group. Then either there exists a p'-element x which is a power of a δ^*_{j-1}-commutator with the property that $[\delta^*_j(G), x]$ is noncyclic, or $\delta^*_j(G)$ is cyclic and $j = 1$.*

Proof. For simplicity denote $\delta^*_j(G)$ by P. Suppose first that P is cyclic. If $j \geqslant 2$, then in view of Lemma 2.4 we deduce that $P = 1$, a contradiction. Hence, if P is cyclic, we have $j = 1$. Now assume that P is noncyclic.

Consider the case where $j = 1$. We wish to show that there exists a p'-element $x \in G$ with the property that $[P, x]$ is noncyclic. Let L be a Hall p'-subgroup in G and suppose that $[P, x]$ is cyclic for every $x \in L$. If $L/C_L(P)$ is not cyclic, we obtain a contradiction with Lemma 2.11. Therefore assume that $L/C_L(P)$ is cyclic. Let a be an element of L such that $\langle a, C_L(P)\rangle = L$. We have $P = [P, L] = [P, a]$, which is again a contradiction since $[P, a]$ is cyclic.

Hence we may assume that $j \geqslant 2$. Moreover we assume that G is a counter-example with minimal possible order. Since $\delta^*_{j+1}(G) = 1$, it follows that G is soluble and the Fitting height precisely $j + 1$. By [12] G possesses a tower of height $j + 1$, i.e., a subgroup $P_0 \dots P_{j-2}P_{j-1}P_j$, where $P_j \leqslant P$. Again P_j is noncyclic and therefore, in view of minimality of $|G|$, we have $G = P_0 \dots P_{j-2}P_{j-1}P_j$ and $P_j = P$.

By [11, Lemma 2.5], each subgroup P_i of the tower is generated by δ^*_{i-1}-commutators contained in P_i. Set $H = P_{j-1}$. We know that $P = [P, H]$. Let B be the set of all elements of H which can be written as powers of δ^*_{j-1}-commutators and assume that $[P, b]$ is cyclic for any b in B. First we consider the case where H has odd order.

Let b_1, b_2 be elements of B and $B_0 = \langle b_1, b_2\rangle$. We have $[P, B_0] = [P, b_1][P, b_2]$. Consider now the subgroup $\Omega_1([P, B_0])$. Obviously, $\Omega_1([P, B_0])$ can be viewed as a linear space of dimension at most two over the field with p elements. It is well-known that the nilpotent subgroups of odd order of $GL(2, p)$ are abelian. Hence, we conclude that the derived group of B_0 centralizes $\Omega_1([P, B_0])$ and, by Lemma 2.1(3), also centralizes P. Recall that B_0 is a subgroup generated by two arbitrarily chosen elements $b_1, b_2 \in B$. By Lemma 2.8 we have $H = \langle B\rangle$, and so we conclude that H' centralizes P. Let $\bar{G} = G/C_G(P)$. There is a natural action of $\bar{G}$ on P and so we will view $\bar{G}$ as a group of automorphisms of P. We already know that $\bar{H}$ is abelian and it is clear that $\delta^*_j(\bar{G}) = 1$.

Suppose first that $\bar{H}$ is cyclic and choose an element $b \in B$ such that $\bar{H}$ is generated by $bC_G(P)$. We have $P = [P, \bar{H}] = [P, b]$ which is cyclic, a contradiction. Hence, $\bar{H}$ is not cyclic. Let q be the prime such that H has q-power order. By induction the group $\bar{G}$ contains a q'-element y which is a power of δ^*_{j-2}-commutator with the property that $[\bar{Q}, y]$ is noncyclic. Moreover, Lemma 2.7(4) shows that $[\bar{Q}, y]$ consists entirely of δ^*_{j-1}-commutators. For any element $t \in [\bar{Q}, y]$ we can choose $b_t \in B$ such that $[P, t] = [P, b_t]$. Therefore $[P, t]$ is cyclic for each $t \in [\bar{H}, y]$. In view of Lemma 2.11 this leads to a contradiction.

Now consider the case where H is a 2-subgroup. In this case the properties of towers listed before the lemma play an important role in our arguments. As before we have $[P, H] = P$ and we wish to show that H contains a δ^*_{j-1}-commutator x with the property that $[P, x]$ is noncyclic. We can pass to the quotient $G/C_H(P)$ and assume that H acts on P faithfully. Choose a δ^*_{j-2}-commutator $b \in P_{j-2}$. Suppose that b normalizes an abelian subgroup A in H. If $[A, b] \neq 1$, then $[A, b]$ is a noncyclic abelian subgroup which, by Lemma 2.7(4), entirely consists of δ^*_{j-1}-commutators. By Lemma 2.11 $[P, x]$ is noncyclic for some $x \in [A, b]$ and we are done. Therefore $[A, b] = 1$ for every abelian subgroup A of H which is normalized by b.

We know that $[h, \underbrace{b, \ldots, b}_{j-1}]$ is a δ^*_{j-1}-commutator for every $h \in H$. Therefore we can choose $a \in H$ such that a and $[a, b]$ are both nontrivial δ^*_{j-1}-commutators. If both a and $[a, b]$ have order 2, then the subgroup $\langle a, [a, b]\rangle$ is abelian and consists of δ^*_{j-1}-commutators. By Lemma 2.11 $[P, x]$ is noncyclic for some $x \in \langle a, [a, b]\rangle$ and we are done. Therefore we can choose $a \in H$ such that a and $[a, b]$ are both nontrivial δ^*_{j-1}-commutators, the element $[a, b]$ being of order four. Since $a^2 \in Z(H)$ and since $[Z(H), b] = 1$, we have $[a^2, b] = 1$. So we have

$$1 = \left[a^2, b\right] = [a, b][a, b]^a$$

and in particular a inverts $[a, b]$. It follows that a normalizes $[P, [a, b]]$ which is a cyclic subgroup. Now consider the action of the subgroup $D = \langle a, [a, b]\rangle$ on $[P, [a, b]]$. By Lemma 2.3

D' centralizes $[P,[a,b]]$. So in particular $[a,b]^2$ is nontrivial and it centralizes the cyclic subgroup $[P,[a,b]]$. Thus we get a contradiction by Lemma 2.2. The proof is now complete. □

Lemma 3.2. *Let p be a prime, j a positive integer and G a group such that $\delta^*_{j+1}(G)=1$. Let P be the Sylow p-subgroup of $\delta^*_j(G)$ and assume that $[P,x]$ is cyclic for every p'-element x which is a power of a δ^*_{j-1}-commutator. Then P is cyclic.*

Proof. By passing to the quotient $G/O_{p'}(\delta^*_j(G))$ we may assume that $\delta^*_j(G)$ is a p-group and that $P=\delta^*_j(G)$. If P is abelian, the result is immediate from Lemma 3.1. Thus, we assume that P is not abelian and use induction on the nilpotency class of P. We consider the quotient $G/Z(P)$ and by induction we conclude that $P/Z(P)$ is cyclic. However this implies that P is abelian and we get a contradiction. □

Lemma 3.3. *Let p be a prime, j a positive integer and G a group such that $\delta^*_{j+1}(G)=1$. Suppose that G possesses m cyclic subgroups whose union contains all δ^*_j-commutators of G and that the Sylow p-subgroup P of $\delta^*_j(G)$ is nilpotent of class c. Let x be a p'-element which is a power of δ^*_{j-1}-commutator in G such that $[P,x]$ is noncyclic. Then the order of $[P,x]$ is (c,m)-bounded.*

Proof. The conjugation by the element x induces a p'-automorphism of P. Since every element of the form $[y,\underbrace{x,\ldots,x}_{j}]$, with $y\in P$ is a δ^*_j-commutator, Lemma 2.10 shows that the order of $[P,x]$ is (c,m)-bounded, as desired. □

Lemma 3.4. *Let j be a non-negative integer and G a group such that $\delta^*_j(G)$ is nilpotent of class c. Suppose that G possesses m cyclic subgroups whose union contains all δ^*_j-commutators of G. Then $\delta^*_j(G)$ contains a subgroup Δ of (c,m)-bounded order which is normal in G and has the property that $\delta^*_j(G)/\Delta$ is cyclic. If $j\geqslant 2$, then $\delta^*_j(G)=\Delta$.*

Proof. We argue by induction on j. Suppose first that $j=0$. In this case G is nilpotent of class c and it is covered by m cyclic subgroups. The result is rather straightforward applying Lemma 2.9 to each Sylow subgroup of G.

So we assume that $j\geqslant 1$. Let P be a Sylow p-subgroup of $\delta^*_j(G)$ for some prime p. Denote by Δ_p the subgroup generated by all subgroups of the form $[P,y]$, where y ranges over the set of all p'-elements which are powers of δ^*_{j-1}-commutators in G such that $[P,y]$ is noncyclic. By Lemma 3.3 the orders of all such subgroups $[P,y]$ have a common bound, which depends only on c and m. We observe that Δ_p is a group which is nilpotent of class at most c and is generated by elements of (c,m)-bounded order. Hence the exponent of Δ_p is (c,m)-bounded. Moreover, by Lemma 2.7(3) Δ_p is generated by δ^*_j-commutators that are all contained in m cyclic subgroups, and so we conclude that Δ_p has at most m generators. It follows that the order of Δ_p is (c,m)-bounded. We further observe that since the bound on the order of Δ_p does not depend on p, it follows that $\Delta_p = 1$ for all primes p which are bigger than certain number depending only on c and m.

Let Δ be the product of the subgroups Δ_p over all prime divisors of $|\delta_j^*(G)|$. It is clear that $|\Delta|$ is (c,m)-bounded. Consider the quotient G/Δ. For simplicity, we just assume that $\Delta = 1$. Then $[P,x]$ is cyclic for every p'-element x which is a power of a δ_{j-1}^*-commutator. Then, by Lemma 3.2, P is cyclic. Thus all Sylow subgroups of $\delta_j^*(G)$ are cyclic. It follows that $\delta_j^*(G)/\Delta$ is cyclic. Of course, if $j \geqslant 2$, then by Lemma 2.4 we have $\delta_j^*(G) = \Delta$. $\square$

We are now ready to complete the proof of Theorem 1.3.

Proof. Recall that $j \geqslant 2$ and G possesses m cyclic subgroups whose union contains all δ_j^*-commutators of G. We wish to show that the order of $\delta_j^*(G)$ is m-bounded. Let $C_1, \dots, C_m$ be the cyclic subgroups whose union contains all δ_j^*-commutators of G. Without loss of generality we assume that each subgroup C_i is generated by δ_j^*-commutators (not necessarily by a single δ_j^*-commutator). Thus, $\delta_j^*(G) = \langle C_1, \dots, C_m\rangle$ and in particular it follows that $\delta_j^*(G)$ can be generated by m elements. Let $x \in G$ be a δ_j^*-commutator. For any $g \in G$ the conjugate x^g is again a δ_j^*-commutator and so $x^g \in C_i$ for some i. Since C_i is cyclic, it contains only at most one subgroup of any given order and we conclude that the cyclic subgroup $\langle x\rangle$ has at most m conjugates. Therefore the index of the normalizer of $\langle x\rangle$ in G is at most m. Let N be the intersection of all normalizers of cyclic subgroups generated by a δ_j^*-commutator and set $K = \delta_j^*(G) \cap N$. Since $\delta_j^*(G)$ is m-generated, it follows that the number of subgroups of $\delta_j^*(G)$ whose index is at most m is m-bounded [6, Theorem 7.2.9] and so we deduce that the index of K in $\delta_j^*(G)$ is m-bounded as well. It is clear that K normalizes each of the subgroups $C_1, \dots, C_m$. This implies that K is nilpotent of class at most 2. Indeed, since $Aut\, C_i$ is abelian for every $i = 1, \dots, m$, we deduce that $K/C_K(C_i)$ is abelian. So K' centralizes $\delta_j^*(G)$ and therefore $K' \leqslant Z(K)$.

Recall that given a group G, the last term of the upper central series of G is called the hypercenter of G. It will be denoted by $Z_\infty(G)$. Let us show that $K \leqslant Z_\infty(\delta_j^*(G))$. Choose a Sylow p-subgroup P of K. It is clear that P is normal in G. If all the subgroups C_i have p-power order, then all δ_j^*-commutators of G are p-elements and by [11, Theorem 2.4] G is soluble and $\delta_j^*(G)$ is a p-subgroup. Thus $\delta_j^*(G)$ is nilpotent and so, we have $Z_\infty(\delta_j^*(G)) = \delta_j^*(G)$ and the inclusion $P \leqslant Z_\infty(\delta_j^*(G))$ is clear. Otherwise, choose a p'-element $x \in C_i$ for some i which is a power of a δ_j^*-commutator. Since P normalizes $\langle x\rangle$, it follows that x centralizes P. Therefore $\delta_j^*(G)/C_{\delta_j^*(G)}(P)$ is a p-group and again the inclusion $P \leqslant Z_\infty(\delta_j^*(G))$ follows. Thus, $P \leqslant Z_\infty(\delta_j^*(G))$ for every prime p and hence indeed $K \leqslant Z_\infty(\delta_j^*(G))$.

Therefore the index of $Z_\infty(\delta_j^*(G))$ in $\delta_j^*(G)$ is m-bounded. Thus, by Baer's Theorem [10, Corollary 2, p. 113], $\gamma_\infty(\delta_j^*(G))$ has m-bounded order. Passing to the quotient $G/\gamma_\infty(\delta_j^*(G))$ we can assume that $\delta_j^*(G)$ is nilpotent. Hence $\delta_j^*(G)$ is the direct product of its Sylow subgroups. It is sufficient to show that any Sylow subgroup of $\delta_j^*(G)$ has bounded order. Let us choose p a prime that divides $|\delta_j^*(G)|$ and pass to the quotient $G/O_{p'}(\delta_j^*(G))$. So we assume that $\delta_j^*(G)$ is a p-group. In view of Lemma 3.4 it is now sufficient to bound the nilpotency class of $\delta_j^*(G)$. It has already been mentioned that K' centralizes $\delta_j^*(G)$ and therefore we can pass to the quotient G/K' and, without loss of generality, assume that K is abelian. Choose generators $x_1, \dots, x_m$ of the subgroups $C_1, \dots, C_m$ and let t be the index of K in $\delta_j^*(G)$. Since each subgroup $K\langle x_i\rangle$ is nilpotent of class at most 2 and since ${x_i}^t \in K$, it follows that K^t centralizes x_i for each $i = 1, \dots, m$. In other words $K^t \leqslant Z(\delta_j^*(G))$. Passing again to the quotient $G/Z(\delta_j^*(G))$ we can assume that $K^t = 1$. Since the index t of K in $\delta_j^*(G)$ is m-bounded and since $\delta_j^*(G)$ can be generated by m elements, we conclude that the minimal number of generators for K is m-bounded. Combining this with the fact that $K^t = 1$, we immediately deduce that the order of K and therefore that of $\delta_j^*(G)$ are m-bounded. Of course, this implies that so is the nilpotency class of $\delta_j^*(G)$. The proof is complete. □

4. Theorem 1.1

In this section we will deal with Theorem 1.1. The proof is similar to that of Theorem 1.3 but in fact it is easier. Therefore we will not give a detailed proof here but rather describe only some steps.

The next lemma is similar to Lemma 3.1.

Lemma 4.1. *Let p be a prime and G a metanilpotent group. Suppose that the Sylow p-subgroup P of $\gamma_2^*(G)$ is abelian and noncyclic. Then there exists a p'-element x with the property that $[P, x]$ is noncyclic.*

Proof. By Lemma 2.5 there is a Hall p'-subgroup H of G such that $P = [P, H]$. Now we consider the quotient $H/C_H(P)$ with acts faithfully on P. If $H/C_H(P)$ is noncyclic, then by Lemma 2.11 there exists an element x in H such that $[P, x]$ is noncyclic. Therefore we assume that $H/C_H(P)$ is cyclic and let x be an element in H such that $xC_H(P)$ generates $H/C_H(P)$. Then $P = [P, x]$ is noncyclic and x is the required element. □

The proof of the next lemma follows word-by-word that of Lemma 3.2. Therefore we omit the details.

Lemma 4.2. *Let p be a prime and G a metanilpotent group. Let P be the Sylow p-subgroup of $\gamma_2^*(G)$ and assume that $[P, x]$ is cyclic for every p'-element x. Then P is cyclic.*

The next results are similar to Lemmas 3.3 and 3.4. Their proofs can be obtained in the same way as those of Lemmas 3.3 and 3.4 with only obvious changes required.

Lemma 4.3. *Let p be a prime, j a positive integer and G a metanilpotent group. Suppose that G possesses m cyclic subgroups whose union contains all γ_j^*-commutators of G, and* *that the Sylow p-subgroup P of $\gamma_j^*(G)$ is nilpotent of class c. Let x be a p'-element in G such that $[P,x]$ is noncyclic. Then the order of $[P,x]$ is (c,m)-bounded.*

Lemma 4.4. *Let j be a positive integer and G a group such that $\gamma_j^*(G)$ is nilpotent of class c. Suppose that G possesses m cyclic subgroups whose union contains all γ_j^*-commutators of G. Then $\gamma_j^*(G)$ contains a subgroup Δ, of (c,m)-bounded order, which is normal in G and has the property that $\gamma_j^*(G)/\Delta$ is cyclic.*

From this we deduce our Theorem 1.1.

Proof of Theorem 1.1. Recall that G has m cyclic subgroups whose union contains all γ_j^*-commutators of G. We wish to prove that $\gamma_j^*(G)$ contains a subgroup Δ, of m-bounded order, which is normal in G and has the property that $\gamma_j^*(G)/\Delta$ is cyclic. Let $C_1,\dots,C_m$ be the cyclic subgroups whose union contains all γ_j^*-commutators of G. We assume that each subgroup C_i is generated by γ_j^*-commutators. Thus, $\gamma_j^*(G)=\langle C_1,\dots,C_m\rangle$ and in particular it follows that $\gamma_j^*(G)$ can be generated by m elements. Let N be the intersection of all normalizers of cyclic subgroups generated by a γ_j^*-commutator and $K=\gamma_j^*(G)\cap N$. Arguing as in the proof of Theorem 1.3 we deduce that the index of K in $\gamma_j^*(G)$ is m-bounded. It is clear that K is nilpotent of class at most 2 and $K\leqslant Z_\infty(\gamma_j^*(G))$. By Baer's Theorem $\gamma_\infty(\gamma_j^*(G))$ has m-bounded order. Passing to the quotient $G/\gamma_\infty(\gamma_j^*(G))$ we can assume that $\gamma_j^*(G)$ is nilpotent and, with further reductions, that $\gamma_j^*(G)$ is a p-group. In view of Lemma 4.4 it is now sufficient to bound the nilpotency class of $\gamma_j^*(G)$. Since K' centralizes $\gamma_j^*(G)$, we can pass to the quotient G/K' and without loss of generality assume that K is abelian. Choose generators $x_1,\dots,x_m$ of the subgroups $C_1,\dots,C_m$ and let t be the index of K in $\gamma_j^*(G)$. Since each subgroup $K\langle x_i\rangle$ is nilpotent of class at most 2 and since ${x_i}^t\in K$, it follows that K^t centralizes x_i for each $i=1,\dots,m$. In other words $K^t\leqslant Z(\gamma_j^*(G))$. Passing again to the quotient $G/Z(\gamma_j^*(G))$ we can assume that $K^t=1$. Since the index t of K in $\gamma_j^*(G)$ is m-bounded and $\gamma_j^*(G)$ can be generated by m elements, we conclude that the minimal number of generators for K is m-bounded. Combining this with the fact that $K^t=1$, we deduce that the order of K and therefore that of $\gamma_j^*(G)$ are m-bounded. Of course, this implies that so is the nilpotency class of $\gamma_j^*(G)$. The proof is complete. □

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