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Commutators and commutator subgroups in profinite groups [☆]



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ABSTRACT

Let G be a profinite group. We prove that the commutator subgroup G' is finite-by-procyclic if and only if the set of all commutators of G is contained in a union of countably many procyclic subgroups.

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1. Introduction

Let G be a profinite group. If G is covered by countably many closed subgroups, then by Baire Category Theorem [11, p. 200] at least one of the subgroups is open. This simple observation suggests that if G is covered by countably many closed subgroups with certain specific properties, then the structure of G is similar to that of the covering subgroups. For example, if G is covered by countably many periodic subgroups, then G

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is locally finite. We recall that the group G is periodic (or torsion) if each element in G has finite order. The group is locally finite if each of its finitely generated subgroups is finite. Following his solution of the restricted Burnside problem [19,20] and using Wilson's reduction theorem [17], Zelmanov proved that periodic compact groups are locally finite [21]. Another example is that if G is covered by countably many subgroups of finite rank, then G has finite rank. The profinite group G is said to have finite rank at most r if each closed subgroup of G can be topologically generated by at most r elements. A somewhat less obvious result of the same nature is that a profinite group is covered by countably many procyclic subgroups if and only if it is finite-by-procyclic (see Proposition 2.12 in the next section). The group G is called finite-by-procyclic if it has a finite normal subgroup N such that G/N is procyclic.

If $x, y \in G$, then $[x, y] = x^{-1}y^{-1}xy$ is the commutator of x and y . The closed subgroup of G generated by all commutators is the commutator subgroup G' of G . In general, elements of G' need not be commutators (see for instance [10] and references therein). On the other hand, Nikolov and Segal showed that for any positive integer m there exists an integer $f(m)$ such that if G is m -generator, then every element in G' is a product of at most $f(m)$ commutators [12]. Several recent results indicate that if the set of all commutators is covered by finitely, or countably, many subgroups with certain specific properties, then the structure of G' is somehow similar to that of the covering subgroups.

It was shown in [2] that if G is a profinite group that has finitely many periodic subgroups (respectively, subgroups of finite rank) whose union contains all commutators, then G' is locally finite (respectively, G' is of finite rank). In [1] similar results were obtained for the case where commutators are covered by countably many subgroups: if G is a profinite group that has countably many periodic subgroups (respectively, subgroups of finite rank) whose union contains all commutators, then G' is locally finite (respectively, G' is of finite rank). In [4] the corresponding results were obtained for profinite groups in which commutators of higher order are covered by countably many periodic subgroups, or subgroups of finite rank. It was shown in [7] that if G is a profinite group that has finitely many, say m , procyclic subgroups whose union contains all commutators, then G' is finite-by-procyclic. In fact, G' has a finite characteristic subgroup M of m -bounded order such that G'/M is procyclic. Moreover, if G is a pro- p group that has m procyclic subgroups whose union contains all commutators, then G' is either finite of m -bounded order or procyclic. Earlier, Fernández-Alcober and Shumyatsky proved that if G is an abstract group in which the set of all commutators is covered by finitely many cyclic subgroups, then the commutator subgroup G' is either finite or cyclic [6].

In the present article we deal with profinite groups in which the commutators are covered by countably many procyclic subgroups. The main result is the following theorem.

Theorem 1.1. *Let G be a profinite group. The commutator subgroup G' is finite-by-procyclic if and only if the set of all commutators of G is contained in a union of countably many procyclic subgroups.*

We notice that unlike in the other examples, the structure of pro- p groups G in which the commutators are covered by finitely many procyclic subgroups is different from that of pro- p groups in which commutators are covered by countably many procyclic subgroups: in the former case G' is either finite or procyclic while, according to [Theorem 1.1](#), this is not necessarily true in the latter case.

An interesting observation that follows from [Theorem 1.1](#) and [Proposition 2.12](#) is that if the set of commutators of a profinite group G is contained in a union of countably many procyclic subgroups, then the whole commutator subgroup G' is contained in a union of countably many procyclic subgroups. But of course we cannot claim that the family of procyclic subgroups that covers the set of all commutators is necessarily the same as the one that covers G' .

Another noteworthy fact that can be deduced from [Theorem 1.1](#) concerns profinite groups G such that G' is pro- p . Assume that G is such a group and the set of commutators of G is contained in a union of countably many procyclic subgroups of G' . Then at least one of the subgroups is open in G' . Indeed, let C_1, C_2, \dots be countably many procyclic subgroups of G' containing the commutators. [Theorem 1.1](#) tells us that G' has a finite normal subgroup N such that G'/N is procyclic. If G' is finite, there is nothing to prove. So we assume that G'/N is infinite. Since G' is a pro- p group, it follows that any infinite subgroup of G' is open. In particular, if G contains a commutator, say $x \in C_k$, of infinite order, then C_k is open in G' . Otherwise, if all commutators in G have finite order, then all of them must belong to N and we conclude that, since N contains all commutators, $N = G'$. In that case G' is finite and the result follows.

We do not know whether the similar phenomenon holds if the covering subgroups are not necessarily procyclic.

We have already mentioned that a finite-by-procyclic profinite group is covered by countably many procyclic subgroups. Thus, the hard part of the proof of [Theorem 1.1](#) is the one where we show that if the set of all commutators of G is contained in a union of countably many procyclic subgroups, then G' is finite-by-procyclic. In the next section we provide some helpful results which are used throughout the paper. We also establish that a profinite group is covered by countably many procyclic subgroups if and only if it is finite-by-procyclic ([Proposition 2.12](#)). In [Sections 3](#) and [4](#) we study profinite groups in which the commutators are covered by countably many procyclic subgroups. [Section 3](#) deals with virtually abelian groups while [Section 4](#) is devoted to the metabelian case. After the virtually abelian and the metabelian cases are dealt with, the proof of [Theorem 1.1](#) becomes easy. It is given in the final section.

2. Preliminaries

For a profinite group G we denote by $\pi(G)$ the set of prime divisors of the orders of finite continuous images of G . If a profinite group G has $\pi(G) \subseteq \pi$, then we say that G is a pro- π group. Recall that Sylow theorems hold for p -Sylow subgroups of a profinite group (see, for example, [[18, Ch. 2](#)]). When dealing with profinite groups we consider

only continuous homomorphisms and quotients by closed normal subgroups. If H is a closed subgroup of G such that $\pi(H) \subseteq \pi$, we say that H is a pro- π subgroup of G , or just a π -subgroup of G . The group G possesses a certain property virtually if it has an open subgroup with that property.

Lemma 2.1. *Let A be an abelian profinite group which is virtually procyclic. Then A is finite-by-procyclic.*

Proof. The lemma is immediate from the fact that a finitely generated profinite abelian group is a direct sum of finitely many procyclic subgroups [15, Theorem 4.3.5]. \square

The next lemma follows from the fact that a direct product of two finite groups of coprime orders is cyclic if and only if both factors are cyclic.

Lemma 2.2. *Let $A = \prod_{i \in I} A_i$ be an abelian profinite group that can be written as a Cartesian product of finite subgroups A_i such that $(|A_i|, |A_j|) = 1$ whenever $i \neq j$. Then the following holds:*

- (i) A is procyclic if and only if A_i is cyclic for each i .
- (ii) A is virtually procyclic if and only if all but finitely many of the subgroups A_i are cyclic.

We will also require the following lemma taken from [7, Lemma 2.2].

Lemma 2.3. *Let H be a characteristic subgroup of a profinite group G . Suppose that H possesses a finite normal subgroup N such that H/N is procyclic. Then G has a finite characteristic subgroup M contained in H such that H/M is procyclic.*

Given subgroups A and B of a group G , we denote by $[A, B]$ the subgroup generated by the set $\{[a, b] \mid a \in A, b \in B\}$.

Lemma 2.4. *Let A and B be two subgroups of a group G such that $[A, B] = 1$. Suppose that x is a commutator in elements of A and y is a commutator in elements of B . Then the element xy is a commutator in G .*

Proof. Let $x = [a_1, a_2]$ for some $a_1, a_2 \in A$ and $y = [b_1, b_2]$ for some $b_1, b_2 \in B$. Since $[A, B] = 1$ we have $[a_1 b_1, a_2 b_2] = [a_1, a_2][b_1, b_2] = xy$ and the result follows. \square

The following result is well-known.

Lemma 2.5. *Let G be a procyclic group faithfully (and continuously) acted on by a group A . Then A is an abelian group.*

In the above lemma the hypothesis that A acts on G continuously is superfluous since any automorphism of a pro-cyclic group is continuous. Now we state a well-known fact about coprime actions on finite groups. As usual, $[G, A, A]$ stands for $[[G, A], A]$.

Lemma 2.6 ([8], Theorem 3.5.6). *Let A and G be finite groups with $(|G|, |A|) = 1$ and suppose that A acts on G . Then we have $[G, A, A] = [G, A]$.*

We will require the following two results taken from Guralnick [9].

Theorem 2.7 ([9], Theorem A). *Let P be a Sylow p -subgroup of a finite group G . If $P \cap G'$ is abelian and can be generated by at most two elements, then $P \cap G'$ consists entirely of commutators.*

Lemma 2.8 ([9], Lemma 2.5). *Suppose $G = \langle x, y, B \rangle$ is a finite group with B an abelian subgroup such that $G' \leq B$ and $[x, y]$ of order n . Then the subset $\{[x, y]^e b \mid b \in [G, B], (e, n) = 1\}$ consists of commutators.*

In the present paper Lemma 2.8 will be used in the following special form.

Lemma 2.9. *Let $A = \langle x, y \rangle$ be a finite abelian 2-generator group acting on a finite abelian group B . Then every element of $[B, A]$ is a commutator.*

Proof. Put $G = BA$. It is clear that $G' \leq B$. So it follows from Lemma 2.8 that $[G, B]$ consists of commutators. In particular, $[B, A]$ consists of commutators. \square

Lemma 2.10. *Let G be a group and N a normal subgroup of G . Set $K = G/C_G(N)$ and consider the natural action of K on N . Then $[N, G] = [N, K]$.*

Proof. If $n \in N$ and $g \in G$, we have $[n, g] = [n, gC_G(N)]$. Thus, the equality $[N, G] = [N, K]$ follows. \square

We denote by $\gamma_\infty(G)$ the intersection of all terms of the lower central series of G . It is clear that a finite group G is nilpotent if and only if $\gamma_\infty(G) = 1$. Therefore a profinite group G is pronilpotent if and only if $\gamma_\infty(G) = 1$. By a well-known property of finite groups $\gamma_\infty(G)$ is generated by all commutators $[x, y]$, where x and y have mutually coprime orders (see for example [13, Theorem 2.1]).

Lemma 2.11. *Let $G = AB$ be a finite group that is a product of two subgroups A and B with $(|A|, |B|) = 1$. Suppose that $p_1 < p_2$ whenever $p_1 \in \pi(A)$ and $p_2 \in \pi(B)$. If $\gamma_\infty(G)$ is cyclic, then B is normal in G .*

Proof. Let G be a counterexample of minimal possible order. Then G is not nilpotent and so $\gamma_\infty(G) \neq 1$. Thus, since $\gamma_\infty(G)$ is cyclic, it follows that G has a normal subgroup

N of prime order p , for some prime p . By induction, the image of B in the quotient group G/N is normal and therefore the subgroup NB is normal in G . If $p \in \pi(B)$, then $N \leq B$ and we have nothing to prove. Suppose that $p \in \pi(A)$. Since p is smaller than any prime divisor of $|B|$, it follows that $[N, B] = 1$, and so $NB = N \times B$. Therefore B is characteristic in NB , hence normal in G and this leads to a contradiction. \square

We will now prove that a profinite group is covered by countably many procyclic subgroups if and only if it is finite-by-procyclic. The structure of procyclic groups is well-known (cf [15, Theorem 2.7.2]). We will use the fact that a procyclic group G such that $|\pi(G)| < \infty$ has only countably many closed subgroups.

Proposition 2.12. *A profinite group G is covered by countably many procyclic subgroups if and only if G is finite-by-procyclic.*

Proof. Suppose first that G has a finite normal subgroup N such that G/N is procyclic. Set $\pi = \pi(N) \cup \pi(\text{Aut}(N))$. Of course, π is a finite set of primes. Let D be the subgroup of G generated by all π -elements and E the subgroup of G generated by all π' -elements. It is clear that $D = O_\pi(G)$ and $E = O_{\pi'}(G)$. Thus $G = D \times E$.

Let $a \in D$ be an element such that aN is a generator of D/N . We know that $\langle a \rangle$ has at most countably many closed subgroups. Let $1 = A_1, A_2, \dots$ be the closed subgroups of $\langle a \rangle$. In each A_i we choose a generator a_i . Let B be any procyclic subgroup in D . There exists $i \geq 1$ such that $BN = A_iN$. Clearly, $B = \langle a_i x \rangle$ for a suitable $x \in N$. Since there are at most countably many pairs (a_i, x) , it follows that D has only countably many procyclic subgroups, say D_1, D_2, \dots . Recall that $G = D \times E$. We now easily deduce that G is covered by countably many procyclic subgroups, each of the form $D_i \times E$. Thus, we proved that if G has a finite normal subgroup N such that G/N is procyclic, then G is covered by countably many procyclic subgroups. Let us now prove the converse.

Assume that G is covered by countably many procyclic subgroups. We wish to show that G is finite-by-procyclic. By [14, Theorem 1.1] G' is finite. We can pass to the quotient G/G' and without loss of generality assume that G is abelian. Then, by Lemma 2.1, G is finite-by-procyclic, as required. The proof is now complete. \square

3. On virtually abelian groups

It is clear that a profinite group G has rank one if and only if G is procyclic. It was shown in [4, Theorem 2] that if all commutators in G are contained in a union of countably many subgroups of finite rank, then the rank of G' is finite.

Let G be, as in Theorem 1.1, a profinite group in which the set of all commutators is contained in a union of countably many procyclic subgroups. In the course of proving Theorem 1.1 we will often use some simple arguments that show that certain subgroups of G can be assumed procyclic. We will now formalize those arguments as follows.

Remark 1. Suppose that G has a subgroup M such that every element of M is a commutator. It follows that the subgroup M is covered by countably many procyclic subgroups. By Baire Category Theorem one of those procyclic subgroups is open. Thus, by [Proposition 2.12](#), M is finite-by-procyclic.

Remark 2. Suppose that G has an abelian normal subgroup A . For every element $x \in G$ the subgroup $[A, x]$ consists entirely of commutators. Therefore, by [Remark 1](#), the subgroup $[A, x]$ is finite-by-procyclic.

Remark 3. Suppose that G has a normal abelian virtually procyclic subgroup V . The set of all torsion elements in V forms a finite characteristic subgroup M such that V/M is procyclic. Then G' is finite-by-procyclic if and only if the commutator subgroup of G/M is finite-by-procyclic. Thus, we can pass to the quotient G/M and, without loss of generality, assume that V is procyclic.

Remark 4. Let G be a profinite group and T be a procyclic subgroup of G . Then G contains a maximal procyclic subgroup S such that $T \leq S$. Indeed, suppose that this is false and write $T = T_1 < T_2 < \dots$, where T_i are procyclic subgroups of G . Let T_0 be the topological closure of $\bigcup_i T_i$. In any finite quotient of G the image of T_0 is cyclic and therefore T_0 is topologically generated by just one element. Hence, T_0 is procyclic and this proves the claim.

We will now deal with the following particular case of [Theorem 1.1](#).

Lemma 3.1. *Let G be a profinite group in which all commutators are contained in a union of countably many procyclic subgroups. If G' is an abelian pro- p subgroup, then G' is finite-by-procyclic.*

Proof. By [\[4, Theorem 2\]](#) G' has finite rank, say r . The lemma will be proved by induction on r . If $r = 1$, then G' is procyclic and there is nothing to prove.

Assume that $r \geq 2$. Since finitely generated abelian profinite groups decompose as direct sums of procyclic subgroups, torsion elements of G' form a finite normal subgroup, say G_0 . We can pass to the quotient G/G_0 and, without loss of generality, assume that G' is torsion-free. Since G' is abelian, [Remark 2](#) shows that the subgroup $[G', x]$ is finite-by-procyclic for all x in G . Since G' is torsion-free it follows that $[G', x]$ is infinite procyclic. This happens for every $x \in G$. We also note that since G is metabelian, the subgroup $[G', x]$ is normal in G .

Choose a maximal normal procyclic subgroup M in G' . Suppose that G'/M is not torsion-free and let N/M be a finite subgroup in G'/M . Since G'/M is abelian of finite rank, every finite subgroup of G'/M is contained in a finite characteristic subgroup. Hence, we can choose N to be normal in G . By [Lemma 2.1](#) N is finite-by-procyclic. Taking into account that G' is torsion-free, we conclude that N is procyclic. Since M was chosen maximal, this leads to a contradiction. Hence, G'/M is torsion-free. Therefore the

rank of G'/M is strictly less than that of G' . By induction, G'/M is finite-by-procyclic. We already know that G'/M is torsion-free. Therefore G'/M is procyclic. By a profinite version of [Theorem 2.7](#) it follows that every element of G' is a commutator. Hence, by [Remark 1](#), G' is finite-by-procyclic. \square

As usual, the Frattini subgroup of a group T is denoted by $\Phi(T)$. In the proof of the next lemma we use the well-known Schur Theorem that if G is a group whose center has finite index, then G' is finite [[16, Theorem 4.12](#)].

Lemma 3.2. *Let G be a profinite group in which all commutators are contained in a union of countably many procyclic subgroups. If G contains an open normal abelian pro- p subgroup A , then G' is finite-by-procyclic.*

Proof. Suppose that the result is false and take a counterexample G with the index $[G : A]$ as small as possible. If $[G : A] = 1$, there is nothing to prove. If $[G : A] = q$, for some prime q , then G is metabelian and the result follows from [Lemma 3.1](#). So we assume that $[G : A]$ is not a prime number.

Assume first that G/A is not simple. Let K/A be a proper non-trivial normal subgroup of G/A . Since G is a counterexample with $[G : A]$ as small as possible, it follows that K' is finite-by-procyclic. [Lemma 2.3](#) tells us that K' has a finite characteristic subgroup N such that K'/N is procyclic. It is clear that G' is finite-by-procyclic if and only if the commutator subgroup of G/N is finite-by-procyclic. Thus, we can pass to the quotient G/N and, without loss of generality, assume that K' is procyclic. Since the group of automorphisms of a procyclic group is abelian, we have $K' \leq Z(G')$. Since A is open, it follows that the Sylow subgroups of G corresponding to primes other than p are finite. Suppose that K is abelian. We pass to the quotient over the (finite) subgroup generated by all p' -elements of K and assume that K is a pro- p group. Since the index $[G : K]$ is smaller than $[G : A]$, by induction we deduce that G' is finite-by-procyclic. Thus, in the case where K is abelian, G' is finite-by-procyclic as required. This shows that G'/K' is finite-by-procyclic. Let L be the minimal normal subgroup of G' such that L/K' is finite and G'/L is procyclic. Since K' is contained in $Z(G')$, it follows from the Schur Theorem that L' is finite. We pass to the quotient over L' and assume that L is abelian. Further, the argument in [Remark 3](#) allows us to assume that L is procyclic, in which case $L \leq Z(G')$. Since also G'/L is procyclic, it follows that G' is abelian. Now the result is immediate from [Lemma 3.1](#). Thus, in the case where G/A is not simple, or simple of prime order, we are done.

Assume that G/A is a non-abelian simple group. Of course, in this case we have $[A, G, G] = [A, G]$. Since A is normal abelian, we apply [Remarks 2 and 3](#) and assume that $[A, x]$ is procyclic for every $x \in G$. Here we use the fact that G contains only finitely many subgroups of the form $[A, x]$. Put $M = [A, G]$. Suppose first that M is procyclic. Then the equality $M = [M, G]$ implies that $M = 1$. Therefore G is central-by-finite and, by Schur's theorem G' is finite. Thus, we may assume that M is of rank $n \geq 2$.

Since G/A acts faithfully on $M/\Phi(M)$, we have an embedding of G/A in $GL(V)$, where $V = M/\Phi(M)$. However, for any p' -element g of $SL(V)$ the dimension of $[M, g]$ must be at least two. This leads to a contradiction since $[A, x]$ is procyclic for every $x \in G$. The proof is now complete. \square

We will now look at the case where the open normal abelian subgroup A is not necessarily a pro- p subgroup. Our immediate goal is to show that $[A, G]$ is finite-by-procyclic. We denote by $O_\pi(G)$ the unique largest normal pro- π subgroup of G .

Lemma 3.3. *Let G be a profinite group in which all commutators are contained in a union of countably many procyclic subgroups. If G contains an open normal abelian subgroup A , then $[A, G]$ is finite-by-procyclic.*

Proof. Choose a prime $p \in \pi(A)$ and let P be the Sylow pro- p subgroup of A . Note that $[P, G]$ is the Sylow pro- p subgroup of $[A, G]$. Moreover, by Lemma 2.10, in the semidirect product of P by $G/C_G(P)$ we have $[P, G] = [P, G/C_G(P)]$. Since A is open in G , the semidirect product of P by $G/C_G(P)$ is a virtually pro- p group and it follows from Lemma 3.2 that $[P, G]$ is finite-by-procyclic. It is now straightforward that for any finite set of primes σ the subgroup $O_\sigma([A, G])$ is finite-by-procyclic.

Let $\pi = \pi(G/A)$. Since π is finite, it follows that $O_\pi([A, G])$ is finite-by-procyclic. We observe that $[A, G] = O_\pi([A, G]) \times O_{\pi'}([A, G])$. Therefore it is sufficient to show that $O_{\pi'}([A, G])$ is finite-by-procyclic. We can pass to the quotient $G/O_\pi([A, G])$ and simply assume that $[A, G] = O_{\pi'}([A, G])$.

Let $K = G/C_G(A)$. Thus, K is a finite group acting on A by automorphisms. Applying the argument in Remarks 2 and 3 to all subgroups of the form $[A, g]$ (there are only finitely many of them) we assume that $[A, g]$ is procyclic for any $g \in K$. Thus, a finite π -group K acts faithfully and continuously on the profinite abelian π' -group A in such a way that $[A, g]$ is procyclic for any $g \in K$. Suppose that $[A, K]$ is not procyclic. Then we can choose a Sylow pro- p subgroup $P \leq [A, K]$ such that P is not procyclic and $P = [P, K]$. By [3, Lemma 2.11] either the quotient group $K/C_K(P)$ is cyclic, or otherwise, there exists an element $g_0 \in K$ such that $[P, g_0]$ is not procyclic. Obviously under our assumptions the latter case is impossible so we conclude that $K/C_K(P)$ is cyclic. Write $K = \langle g, C_K(P) \rangle$ for some element $g \in K$. Then we have $P = [P, K] = [P, g]$. Since $[P, g]$ is procyclic, this is a contradiction. \square

Now we are ready to prove the main result of this section.

Theorem 3.4. *Let G be a profinite group in which all commutators are contained in a union of countably many procyclic subgroups. If G contains an open normal abelian subgroup A , then G' is finite-by-procyclic.*

Proof. We know from Lemma 3.3 that $[A, G]$ is finite-by-procyclic. We use Remark 3 and assume that $[A, G]$ is procyclic. It follows from Lemma 2.5 that $[A, G] \leq Z(G')$. In the

quotient group $G/[A, G]$ the subgroup A is central so, by Schur’s theorem, $G/[A, G]$ is finite and hence G' is central-by-finite. In particular, G'' is finite. We pass to the quotient over G'' and assume that G' is abelian. Since G' is virtually procyclic, it follows from Lemma 2.1 that G' is finite-by-procyclic. The proof is complete. \square

4. On metabelian groups

The purpose of the present section is to prove the following theorem.

Theorem 4.1. *If G is a metabelian profinite group in which all commutators are contained in a union of countably many procyclic subgroups, then G' is finite-by-procyclic.*

The special case where G' is an abelian pro- p subgroup was already proved in Lemma 3.1. The general case, where G' is not necessarily pro- p , is more complicated since it does not reduce easily to the situation where G' is torsion-free. The case of the theorem where G' is torsion-free is quite easy. Indeed, for any $p \in \pi(G')$ we consider the quotient group $G/O_{p'}(G')$. In view of Lemma 3.1 we conclude that each Sylow p -subgroup of G' is procyclic. Hence, in the case where G' is torsion-free G' is procyclic.

Proof of Theorem 4.1. We start with an easy observation that any Sylow subgroup of G' is finite-by-procyclic. Indeed, choose a prime $p \in \pi(G')$ and consider the quotient $G/O_{p'}(G')$. The commutator subgroup of $G/O_{p'}(G')$ is a pro- p group and therefore, by Lemma 3.1, it is finite-by-procyclic. Thus, the Sylow p -subgroup of G' is finite-by-procyclic.

Let C_1, C_2, \dots , be the countably many procyclic subgroups of G' whose union contains all commutators. Let $x \in G$. By Remark 2 $[G', x]$ is virtually procyclic and consists of commutators. Therefore for any $x \in G$ there exist positive numbers $n(x)$ and $i(x)$ such that $[G', x]^{n(x)} \leq C_{i(x)}$.

For each pair $\alpha = (n, i)$ we define the set

$$S_\alpha = \{x \in G \mid [G', x]^n \leq C_i\}.$$

The sets S_α are closed in G . Indeed, fix a pair $\alpha = (n, i)$ and suppose that $x \notin S_\alpha$. It follows that $[G', x]$ contains an element y such that $y^n \notin C_i$. We can choose an open normal subgroup N in G such that the image of y^n in G/N is not contained in the image of C_i . We see that no element that belongs to the coset xN is contained in S_α . Therefore the complement of S_α is open (for each element in the complement there exists a neighborhood of that element which is entirely contained in the complement). Therefore the set S_α is closed.

It is clear that the group G is covered by the sets S_α . By Baire’s Category Theorem at least one of these sets contains a non-empty interior. Thus, there exists a certain pair (\bar{n}, \bar{i}) such that G possesses an element b and an open normal subgroup H with

the property that $[G', x]^{\bar{\pi}} \leq C_{\bar{\tau}}$ for any $x \in bH$. Let π^* be the set of prime divisors of $\bar{\pi}$. Since G' is abelian of finite rank [4, Theorem 2], the subgroup generated by all π^* -elements of finite order in G' is finite. Passing to the quotient over this subgroup we can assume that for any prime $p \in \pi^*$ the Sylow p -subgroup of G' is infinite procyclic. If necessary, we enlarge the subgroup $C_{\bar{\tau}}$ replacing it by the product of $C_{\bar{\tau}}$ with all Sylow subgroups of G' corresponding to the primes in π^* . Obviously this product is again a procyclic subgroup. To avoid changing the notation we simply assume that $C_{\bar{\tau}}$ contains the Sylow subgroups corresponding to the primes in π^* . It follows that $[G', x] \leq C_{\bar{\tau}}$ for any $x \in bH$. Taking into account that each subgroup of the form $[G', x]$ is normal in G we deduce that $[G', h] \leq [G', bh][G', b] \leq C_{\bar{\tau}}$ for any $h \in H$. Hence, $[G', H] \leq C_{\bar{\tau}}$. Let us prove the following claim.

Claim 1. The third term of the lower central series of G (denoted by $\gamma_3(G)$) is virtually procyclic.

Let $\pi(G) = \{p_1, p_2, \dots\}$ and $\{G_1, G_2, \dots\}$ be a Sylow system in G such that G_i is a Sylow p_i -subgroup of G . Thus, $G_i G_j = G_j G_i$ for all i, j . For every $i = 1, 2, \dots$ put $R_i = \gamma_3(G) \cap G_i$. Since $\Phi(\gamma_3(G)) = \prod_i \Phi(R_i)$, it follows from Lemma 2.2(ii) that $\gamma_3(G)/\Phi(\gamma_3(G))$ is virtually procyclic if and only if $R_i/\Phi(R_i)$ is cyclic for all but finitely many primes p_i . Therefore, since the Sylow subgroups of G' are finite-by-procyclic, if $\gamma_3(G)/\Phi(\gamma_3(G))$ is virtually procyclic then so is $\gamma_3(G)$.

Assume by absurdum that $\gamma_3(G)$ is not virtually procyclic. Pass to the quotient over $\Phi(\gamma_3(G))$ and assume that every subgroup R_i is elementary abelian. Further, each subgroup R_i is finite since G' has finite rank.

Let $G = \langle H, b_1, \dots, b_s \rangle$. Set $K_0 = H, K_1 = \langle H, b_1 \rangle, \dots, K_s = \langle H, b_1, \dots, b_s \rangle = G$. Let \hat{s} be the minimal index in $\{0, \dots, s\}$ for which $[G', K_{\hat{s}}]$ is not virtually procyclic. Since $[G', K_0] = [G', H] \leq C_{\bar{\tau}}$, it is clear that $1 \leq \hat{s} \leq s$. Since all subgroups $[G', b_j]$ consist of commutators, they are virtually procyclic. Using Remark 3 we assume that each subgroup $[G', b_j]$ is procyclic. For the same reason we can assume that $[G', K_{\hat{s}-1}]$ is procyclic.

Let σ be the set of all primes for which the corresponding Sylow subgroups of $[G', K_{\hat{s}}]$ are not cyclic. Since $[G', K_{\hat{s}}]$ is not virtually procyclic, it follows that the set σ is infinite. Moreover $\gamma_3(G/O_{\sigma'}(\gamma_3(G)))$ is not virtually procyclic. Consider the quotient $G/O_{\sigma'}(\gamma_3(G))$ in place of G and just assume that $\sigma = \pi(\gamma_3(G))$. Since $[G', K_{\hat{s}}] = [G', K_{\hat{s}-1}][G', b_{\hat{s}}]$ and since both subgroups $[G', K_{\hat{s}-1}]$ and $[G', b_{\hat{s}}]$ are procyclic while none of the Sylow subgroups of $[G', K_{\hat{s}}]$ is cyclic, we conclude that $[G', K_{\hat{s}-1}] \cap [G', b_{\hat{s}}] = 1$ and $\pi([G', K_{\hat{s}-1}]) = \pi([G', b_{\hat{s}}]) = \sigma$.

Suppose that $[G', h]$ is finite for any $h \in H$. For any positive integer λ define

$$S_{\lambda} = \{h \in H \mid |[G', h]| \leq \lambda\}.$$

The sets S_{λ} cover the subgroup H and it is clear that each set S_{λ} is closed. By Baire's Category Theorem at least one of these sets contains a non-empty interior. Hence there

exists an integer m such that H contains a open normal subgroup H_1 with the property that $|[G', h]| \leq m$ for any $h \in H_1$. Since G' is abelian of finite rank, the subgroup generated by all elements of finite order at most m in G' is finite. We pass to the quotient over this subgroup and thus assume that $H_1 \leq C_G(G')$. In that case $G/C_G(G')$ is finite. The semidirect product of G' by $G/C_G(G')$ is virtually abelian. Lemma 2.10 tells us that $[G', G] = [G', G/C_G(G')]$. By Theorem 3.4 the commutator subgroup of the semidirect product of G' by $G/C_G(G')$ is finite-by-procyclic. Since $\gamma_3(G) = [G', G/C_G(G')]$, we now conclude that $\gamma_3(G)$ is finite-by-procyclic. This is a contradiction since we have assumed that $\gamma_3(G)$ is not virtually procyclic. Hence there exists $h_0 \in H$ such that $[G', h_0]$ is infinite.

Set $D = \langle b_{\hat{s}}, h_0 \rangle$. Lemma 2.10 shows that $[G', D] = [G', D/C_D(G')]$. Furthermore D' centralizes G' and so $D/C_D(G')$ is an abelian 2-generator group. By Lemma 2.9, we deduce that every element of $[G', D]$ is a commutator. By Remark 1 we know that $[G', D]$ is virtually procyclic. An application of Remark 3 allows us to assume that $[G', D]$ is procyclic.

Recall that $[G', b_{\hat{s}}] \leq [G', D]$. Since $\pi([G', b_{\hat{s}}]) = \sigma$, it follows that $[G', b_{\hat{s}}] = [G', D]$. In particular $[G', h_0] \leq [G', b_{\hat{s}}]$. Taking into account that $h_0 \in K_{\hat{s}-1}$, we deduce that the infinite subgroup $[G', h_0]$ is contained in $[G', K_{\hat{s}-1}]$. Therefore $[G', h_0]$ is contained in the intersection of $[G', K_{\hat{s}-1}]$ and $[G', b_{\hat{s}}]$. We have already remarked that the intersection is trivial. It follows that $[G', h_0] = 1$. This is a contradiction. The proof of Claim 1 is complete.

Now using Remark 3 we may take the following assumption.

Additional assumption 1. $\gamma_3(G)$ is procyclic.

Recall that $\{G_1, G_2, \dots\}$ is a Sylow system in G such that G_i is a Sylow p_i -subgroup of G and $G_i G_j = G_j G_i$. From now on we assume that $p_i < p_j$, whenever $i < j$. By a profinite version of Lemma 2.11 G_i normalizes G_j , whenever $i < j$. For every $i = 1, 2, \dots$ set $P_i = G' \cap G_i$. Since $\Phi(G') = \prod_i \Phi(P_i)$, it follows from Lemma 2.2(ii) that $G'/\Phi(G')$ is virtually procyclic if, and only if, $P_i/\Phi(P_i)$ is cyclic for all but finitely many primes p_i . Therefore, since the Sylow subgroups of G' are finite-by-procyclic, $G'/\Phi(G')$ is virtually procyclic if and only if G' is virtually procyclic. Hence, we can pass to the quotient over $\Phi(G')$ and assume that each subgroup P_i is elementary abelian. Thus, from now on our task is to establish the following fact.

Claim 2. All but finitely many subgroups P_i are cyclic.

Suppose that this is false. Each subgroup P_i is finite since G' is of finite rank. Whenever the rank of P_i is at least three we choose a subgroup $M_i \leq P_i$ which is normal in G and satisfies the condition that P_i/M_i has rank two. Such a subgroup M_i does exist because, by Additional Assumption 1, $\gamma_3(G)$ is procyclic. Observe that after passing to the quotient over the Cartesian product of all such M_i , we still have a counterexample to Claim 2. Thus we can make the following

Additional assumption 2. The rank of each P_i is at most two.

By [Theorem 2.7](#), every element in each P_i is a commutator. However we cannot claim that G' consists of commutators.

Claim 3. If G is pronilpotent, then G' is finite-by-procyclic.

Indeed, if G is pronilpotent, then G is the Cartesian product of the subgroups G_i and so G' is the Cartesian product of the subgroups P_i . Combining the fact that every element in each P_i is a commutator in elements of G_i with [Lemma 2.4](#), we deduce that every element of G' is a commutator. Thus, by [Remark 1](#), G' is finite-by-procyclic, as claimed. This completes the proof of [Claim 3](#).

Recall that $\gamma_\infty(G)$ stands for the intersection of all terms of the lower central series of G . We know from [Claim 3](#) that the commutator subgroup of $G/\gamma_\infty(G)$ is finite-by-procyclic. Hence, if $\gamma_\infty(G)$ is finite, then G' is finite-by-procyclic. Therefore, without loss of generality, we may make the following

Additional assumption 3. The subgroup $\gamma_\infty(G)$ is infinite.

By [Claim 3](#) the quotient $G'/\gamma_\infty(G)$ is finite-by-procyclic. Let T be the largest subgroup in G such that $\gamma_\infty(G) \leq T \leq G'$ and $\gamma_\infty(G)$ is open in T . In view of [Remark 3](#) we can assume that T is procyclic. Let τ be the set $\pi(T) \setminus \pi(\gamma_\infty(G))$. Then $O_\tau(G')$ is a finite characteristic subgroup in G . We pass to the quotient $G/O_\tau(G')$ and simply assume that $T = \gamma_\infty(G)$. Thus, we conclude that $G'/\gamma_\infty(G)$ is procyclic. Since every Sylow subgroup P_i is elementary abelian of rank at most two, whenever P_i is non-cyclic we have $P_i \cap \gamma_\infty(G) \neq 1$.

For every $i \geq 2$ set $H_i = \prod_{j < i} G_j$. We know that H_i normalizes G_i and it is clear that $p_i \notin \pi(H_i)$. Observe that $P_i = [G_i, H_i G_i]$. By a profinite version of [Theorem 5.3.5](#) of [\[8\]](#), we have $G_i = C_{G_i}(H_i)[G_i, H_i]$. Further, since $p_i \notin \pi(H_i)$, it follows that $[G_i, H_i]$ is contained in $\gamma_\infty(G)$. The latter is procyclic and therefore $[G_i, H_i]$ is of prime order p_i . We now deduce that $G_i = C_{G_i}(H_i) \times [G_i, H_i]$. Note that whenever P_i is non-cyclic the subgroup $C_{G_i}(H_i)$ must be non-abelian. Otherwise we would have $P_i = [G_i, H_i] \leq \gamma_3(G)$, which leads to a contradiction with [Additional Assumption 1](#). Thus, for any i such that P_i is non-cyclic there exist non-commuting elements $a_i, b_i \in C_{G_i}(H_i)$.

Let X be the set of all commutators in G . The set X is closed [\[5, Ex. 6, Chap. 1\]](#) and it is equipped with the topology inherited from G . The set X is covered by countably many closed subsets $C_i \cap X$ and, by Baire’s Category Theorem, at least one of those subsets contains a non-empty interior. Thus there exist a positive integer \bar{j} , an open subgroup $N \leq G$ and $x \in X$, with $x = [a, b]$ for some elements $a, b \in G$, such that $X \cap xN \leq C_{\bar{j}}$. Of course, the subgroup N can be taken normal. Thus all commutators contained in the coset $[a, b]N$ lie in $C_{\bar{j}}$.

Let us denote by L the product of all G_i for which $p_i \leq |G/N|$ and by J the product of all G_i for which $p_i > |G/N|$. Since G_i normalizes G_j , whenever $i < j$, it follows that J

is normalized by L . Moreover since J is the product of Sylow p_i -subgroups G_i for which p_i does not divide $|G/N|$, we have $J \leq N$. Since $G = LN$, without loss of generality, we can assume that $x = [a, b]$, with $a, b \in L$.

Set

$$I = \{i \mid p_i > |G/N| \text{ and } P_i \text{ is not cyclic}\}.$$

We can assume that the set I is infinite, since otherwise we would have all but finitely many P_i cyclic, as required.

Note that the subgroup $\langle a, b \rangle$ commutes with the subgroup $\langle a_i, b_i \rangle$ for any non-commuting elements $a_i, b_i \in C_{G_i}(H_i)$ and any $i \in I$. This is because a_i and b_i are taken in $C_{G_i}(H_i)$, and so in particular they centralize a, b which belong to $L \leq H_i$. It follows from Lemma 2.4 that the elements of the form $[a, b][a_i, b_i]$ are commutators and so they all lie in $C_{\bar{J}}$. In particular, all commutators $[a_i, b_i]$ lie in $C_{\bar{J}}$, since so does $[a, b]$.

For any $g \in G$ we have $[a, gb] = [a, b][a, g]^b$. Hence all elements of the form $[a, b][a, g]^b$ are commutators. Since all commutators contained in the coset $[a, b]N$ lie in $C_{\bar{J}}$, it follows that $[a, N]^b \leq C_{\bar{J}}$.

Now suppose that $g \in G_i$ for some $i \in I$. Both a and b normalize G_i since L does. On the other hand, a and g are elements of coprime orders and therefore $[a, g]^b$ lies in $\gamma_\infty(G)$. It follows that $[a, g]^b \in \gamma_3(G) \cap G_i$. If $[a, g] \neq 1$, then $P_i = \langle [a, g], [a_i, b_i] \rangle$, where again a_i and b_i are arbitrary non-commuting elements from $C_{G_i}(H_i)$. Indeed, both $[a, g]$ and $[a_i, b_i]$ belong to P_i . Observe that they cannot be in the same cyclic subgroup of P_i . This is because $[a_i, b_i] \in C_{G_i}(H_i)$ while $[a, g] \in [G_i, H_i]$ and we know that $G_i = C_{G_i}(H_i) \times [G_i, H_i]$. By Additional Assumption 2 the subgroup P_i is of rank at most 2. Due to the choice of i the subgroup P_i is non-cyclic. Therefore $P_i = \langle [a, g], [a_i, b_i] \rangle$, as desired. However this is a contradiction since both $[a, g], [a_i, b_i]$ belong to $C_{\bar{J}}$. Thus, we conclude that a commutes with G_i whenever $i \in I$.

Similarly, for any $g \in G$ we have $[ga, b] = [g, b]^a[a, b]$. Hence all elements of the form $[g, b]^a[a, b]$ are commutators and we have $[N, b]^a \leq C_{\bar{J}}$. Thus, arguing as in the preceding paragraph we deduce that b commutes with G_i whenever $i \in I$.

For any $c \in C_L(a)$ we have $[a, cb] = [a, b]$. Thus, taking in the above argument the element cb in place of b , we conclude that cb commutes with G_i whenever $i \in I$. This holds for each $c \in C_L(a)$ and therefore $C_L(a)$ commutes with G_i whenever $i \in I$. Observe that L' is finite, because it is the product of finitely many finite Sylow subgroups of G' . It follows that $C_L(a)$ has finite index in L and so, being closed, $C_L(a)$ must be open in L . Now it is easy to see that L contains an open normal subgroup K which commutes with G_i whenever $i \in I$.

Now let the indices i, j satisfy the condition that $|G/N| < p_i < p_j$. Then $G_i G_j \leq J$ and G_i normalizes G_j . Assume additionally that P_j is not cyclic and suppose that $[G_i, G_j] \neq 1$. If $y \in [G_i, G_j]$, then y can be written as a commutator $[y_1, y_2]$, with $y_1 \in G_i$ and $y_2 \in [G_i, G_j]$. This is because by Lemma 2.6 we have $[G_j, G_i, G_i] = [G_j, G_i]$. Recall that $G_i = C_{G_i}(H_i) \times [G_i, H_i]$. Since $[G_i, H_i]$ is contained in $O_{p_i}(G)$, it follows

that $[G_i, H_i]$ commutes with G_j . Therefore the element y_1 can be chosen in $C_{G_i}(H_i)$. We deduce that the subgroup $\langle a, b \rangle$ commutes with the subgroup $\langle y_1, y_2 \rangle$. Indeed, on the one hand, $y_1 \in C_{G_i}(H_i)$ and so it centralizes a and b . On the other hand, it has been shown above that both a and b commute with all element of G_j whenever $j \in I$. In particular both a and b commute with y_2 . It follows from Lemma 2.4 that the element $[a, b]y$ is a commutator. In view of the fact that all commutators contained in the coset $[a, b]N$ lie in $C_{\bar{J}}$, we conclude that $y \in C_{\bar{J}}$. If $y \neq 1$, then $P_j = \langle [a_j, b_j], y \rangle$. Indeed both $[a_j, b_j]$ and $[y_1, y_2]$ lie in P_j . These elements do not belong to the same cyclic subgroup since $[a_j, b_j]$ is in $C_{G_j}(H_j)$ and $y = [y_1, y_2]$ lies in $[G_j, H_j]$ (we use here that $y_1 \in H_j$ and $y_2 \in G_j$). Moreover by Additional Assumption 2 the subgroup P_j is of rank at most 2 and we are assuming that P_j is non-cyclic. Therefore $P_j = \langle [a_j, b_j], y \rangle$. This leads to a contradiction since we know that both $[a_j, b_j]$ and y lie in $C_{\bar{J}}$. Thus, we have $[G_i, G_j] = 1$ whenever the indices i and j satisfy the conditions that $|G/N| < p_i < p_j$ and P_j is non-cyclic.

Let us show that for every $j \in I$ there exists $i < j$ such that $[G_i, G_j] \neq 1$. Suppose that $[G_i, G_j] = 1$ for all $i < j$ and put $q = p_j$. Then $G/O_{q'}(G')$ is pronilpotent. It follows that $\gamma_\infty(G) \leq O_{q'}(G')$. This contradicts the assumption that $P_j \cap \gamma_\infty(G) \neq 1$ whenever P_j is non-cyclic. Thus, indeed for every $j \in I$ there exists $i < j$ such that $[G_i, G_j] \neq 1$.

On the other hand, $[G_i, G_j] = 1$ whenever the indices i and j satisfy the conditions that $|G/N| < p_i < p_j$ and P_j is non-cyclic. The conclusion is that if $j \in I$ and $[G_i, G_j] \neq 1$ for some $i < j$, then $p_i \leq |G/N|$. It follows that $[L, G_j] \neq 1$ for every $j \in I$.

Recall that K has been taken as an open normal subgroup of L which commutes with G_i whenever $p_i > |G/N|$ and P_i is non-cyclic. Write $L = \langle l_1, \dots, l_t, K \rangle$ for some $l_1, \dots, l_t \in L$. For every $i \in I$ there exists an index $j \in \{1, \dots, t\}$, depending on i , such that l_j acts non-trivially on P_i , and so $G_i \cap \gamma_3(G) = [G_i, l_j]$. Therefore for some $k \in \{1, \dots, t\}$ the set

$$E = \{i \in I \mid [G_i, l_k] \neq 1\}$$

must be infinite. Note that for each $i \in E$, every element in $[G_i, l_k]$ can be written as a commutator $[g_i, l_k]$, where $g_i \in [G_i, l_k]$. In fact, for each $i \in E$ and for any non-commuting elements $a_i, b_i \in C_{G_i}(H_i)$ the subgroup $\langle a_i, b_i \rangle$ commutes with the subgroup $\langle g_i, l_k \rangle$. Indeed, we have $G_i = C_{G_i}(H_i) \times [G_i, H_i]$. Both elements a_i and b_i are in $C_{G_i}(H_i)$ and so they commute with $l_k \in L$. Moreover a_i and b_i commute with g_i because $g_i \in [G_i, H_i]$. Therefore by Lemma 2.4 the product $[a_i, b_i][g_i, l_k]$ is again a commutator. We are ready to prove the following claim.

Claim 4. Every element of the Cartesian product $\prod_{i \in E} P_i$ is a commutator.

Let $x = \prod_{i \in E} x_i$, where x_i belongs to P_i . For any i the element x_i can be written as $[a_i, b_i][g_i, l_k]$ for some $g_i \in [G_i, l_k]$ and some non-commuting elements $a_i, b_i \in C_{G_i}(H_i)$. Moreover for any $i_1, i_2 \in E$, the subgroup $\langle a_{i_1}, b_{i_1} \rangle$ commutes with the subgroups $\langle a_{i_2}, b_{i_2} \rangle$, $\langle g_{i_1}, l_k \rangle$ and $\langle g_{i_2}, l_k \rangle$. Hence we have $[g_{i_1}, l_k][g_{i_2}, l_k] = [g_{i_1}g_{i_2}, l_k]$. Write

$$g_0 = \prod_{i \in E} g_i, \quad a_0 = \prod_{i \in E} a_i, \quad \text{and} \quad b_0 = \prod_{i \in E} b_i.$$

Then we have

$$x = [a_{i_1}, b_{i_1}][a_{i_2}, b_{i_2}] \cdots [g_{i_1} g_{i_2} \cdots, l_k] = [a_0, b_0][g_0, l_k].$$

Since also the subgroup $\langle g_0, l_k \rangle$ commutes with the subgroup $\langle a_0, b_0 \rangle$, the element x can be written as $[a_0 g_0, b_0 l_k]$, and so it is a commutator, as claimed. The proof of Claim 4 is complete.

Since every element in $\prod_{i \in E} P_i$ is a commutator, we conclude that $\prod_{i \in E} P_i$ is virtually procyclic. Obviously we now have a contradiction since P_i is not cyclic whenever $i \in E$. This concludes the proof of Claim 2. By Lemma 2.2 G' is virtually procyclic. Lemma 2.1 implies that G' is finite-by-procyclic. The proof of the theorem is now complete. \square

5. Main result

The proof of Theorem 1.1 will now be fairly easy.

Proof of Theorem 1.1. We wish to prove that if G is a profinite group, then G' is finite-by-procyclic if and only if the set of all commutators of G is contained in a union of countably many procyclic subgroups.

So assume that the set of all commutators in G is contained in a union of countably many procyclic subgroups. It follows from Theorem 7 of [1] that G' is virtually soluble. Of course, this implies that G is virtually soluble. Choose a normal soluble open subgroup N in G . Let d be the derived length of N and argue by induction on d . If $d \leq 1$, then the result follows from Theorem 3.4. Hence we can assume that $d \geq 2$. Let M be the last non-trivial term of the derived series of N . By induction G'/M is finite-by-procyclic. Let T be the metabelian term of the derived series of N . Then $M = T'$ and so, by Theorem 4.1, M is finite-by-procyclic. Lemma 2.3 now tells us that M has a finite characteristic subgroup M_0 such that M/M_0 is procyclic. We can pass to the quotient G/M_0 and without loss of generality assume that M is procyclic. Then by Lemma 2.5 we have $M \leq Z(G')$. Recall that G'/M is finite-by-procyclic. Applying Lemma 2.3 to the quotient group G/M , we obtain that G' contains a characteristic subgroup K such that K/M is finite and G'/K is procyclic. Since $M \leq Z(G')$, by Schur's theorem K' is finite. We pass to the quotient G/K' and without loss of generality assume that K is abelian. So K is an abelian virtually procyclic subgroup. In view of Remark 3 we can assume that K is procyclic. Then, again by Lemma 2.5, $K \leq Z(G')$. Since G'/K is procyclic, it follows that G is metabelian and the result follows from Theorem 4.1.

Thus, we proved that if the set of all commutators in G is contained in a union of countably many procyclic subgroups, then G' is finite-by-procyclic. The converse is immediate from Proposition 2.12. The proof is now complete. \square

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